FPT algorithmic techniques

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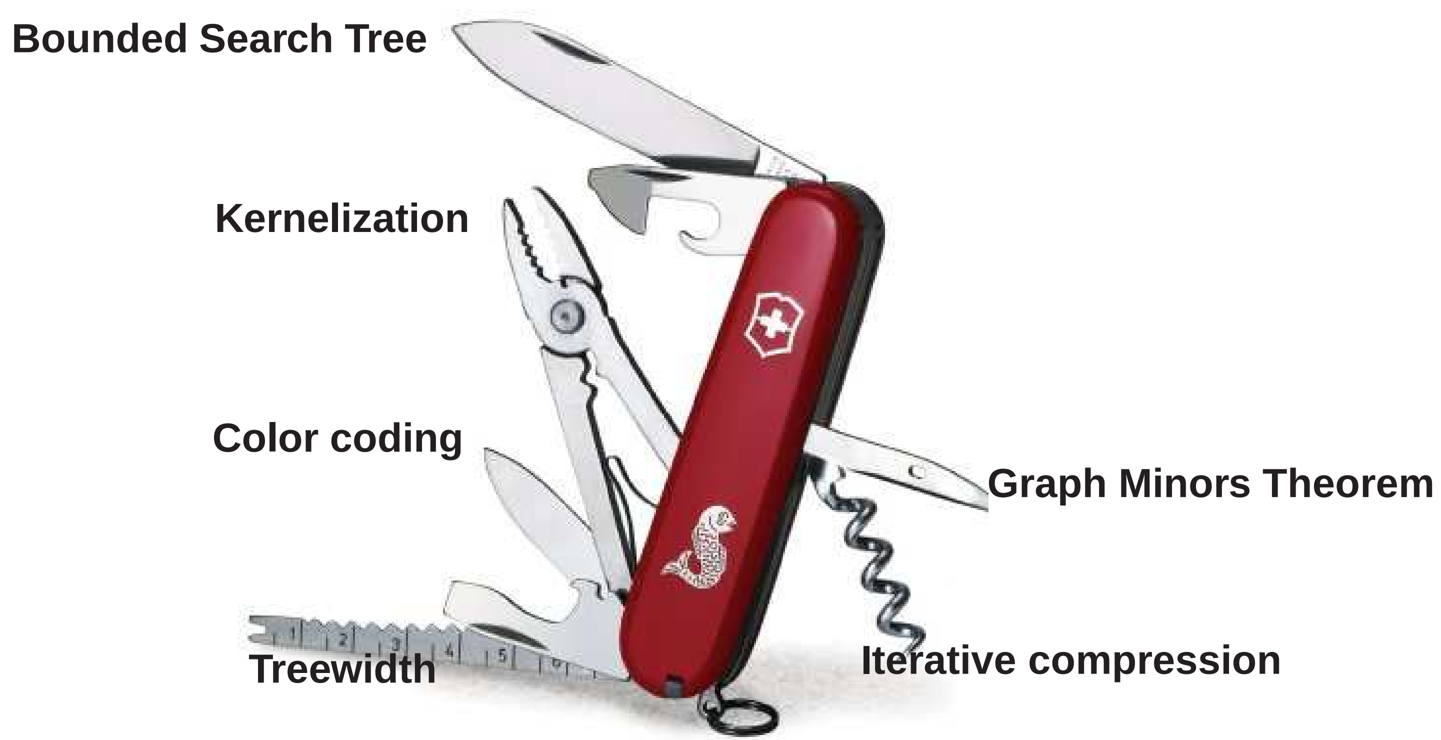
AGAPE'09 Spring School on Fixed Parameter and Exact Algorithms

May 25-26, 2009, Lozari, Corsica (France)

FPT algorithmic techniques

- Significant advances in the past 20 years or so (especially in recent years).
- Powerful toolbox for designing FPT algorithms:



Goals

- Demonstrate techniques that were successfully used in the analysis of parameterized problems.
- There are two goals:
  - Determine quickly if a problem is FPT.
  - Design fast algorithms.
- Warning: The results presented for particular problems are not necessarily the best known results or the most useful approaches for these problems.
- Conventions:
  - Unless noted otherwise, k is the parameter.
  - O^* notation: $O^*(f(k))$ means $O(f(k) \cdot n^c)$ for some constant c.
  - Citations are mostly omitted (only for classical results).
  - We gloss over the difference between decision and search problems.

Kernelization

Kernelization

Definition: **Kernelization** is a polynomial-time transformation that maps an instance (I, k) to an instance (I', k') such that

- (I, k) is a yes-instance if and only if (I', k') is a yes-instance,
- $k' \le k$, and
- $|I'| \le f(k)$ for some function $f(k)$.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.
Proof: Solve the instance (I', k') by brute force.

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Simple fact: If a problem has a kernelization algorithm, then it is FPT.
Proof: Solve the instance (I',k') by brute force.

Converse: Every FPT problem has a kernelization algorithm.
Proof: Suppose there is an $f(k)n^c$ algorithm for the problem.

- If $f(k) \leq n$, then solve the instance in time $f(k)n^c \leq n^{c+1}$, and output a trivial yes- or no-instance.
- If $n < f(k)$, then we are done: a kernel of size $f(k)$ is obtained.

Kernelization for VERTEX COVER

General strategy: We devise a list of reduction rules, and show that if none of the rules can be applied and the size of the instance is still larger than $f(k)$, then the answer is trivial.

Reduction rules for VERTEX COVER instance (G, k):

Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$
Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$

If neither Rule 1 nor Rule 2 can be applied:

- If $|V(G)| > k(k+1) \Rightarrow$ There is no solution (every vertex should be the neighbor of at least one vertex of the cover).
- Otherwise, $|V(G)| \leq k(k+1)$ and we have a $k(k+1)$ vertex kernel.

Kernelization for Vertex Cover

Let us add a third rule:

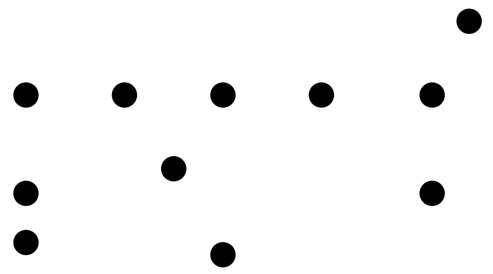
Rule 1: If v is an isolated vertex $\Rightarrow (G \setminus v, k)$
Rule 2: If $d(v) > k \Rightarrow (G \setminus v, k-1)$
Rule 3: If $d(v) = 1$, then we can assume that its neighbor u is in the solution $\Rightarrow (G \setminus (u \cup v), k-1)$.

If none of the rules can be applied, then every vertex has degree at least 2.
$\Rightarrow |V(G)| \leq |E(G)|$

- If $|E(G)| > k^2 \Rightarrow$ There is no solution (each vertex of the solution can cover at most k edges).
- Otherwise, $|V(G)| \leq |E(G)| \leq k^2$ and we have a k^2 vertex kernel.

Covering Points with Lines

Task: Given a set P of n points in the plane and an integer k, find k lines that cover all the points.



Note: We can assume that every line of the solution covers at least 2 points, thus there are at most n^2 candidate lines.

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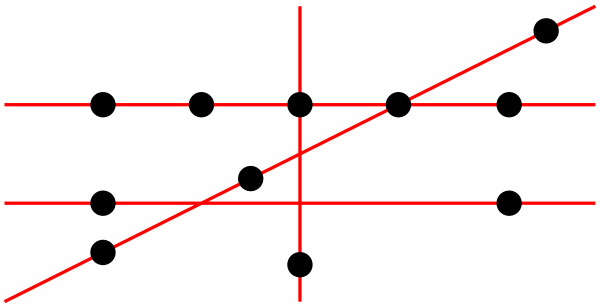

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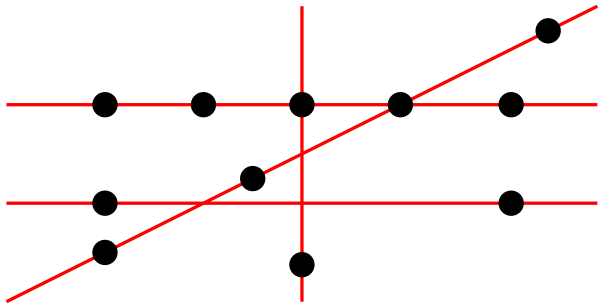


Note: We can assume that every line of the solution covers at least 2 points, thus there are at most n^2 candidate lines.

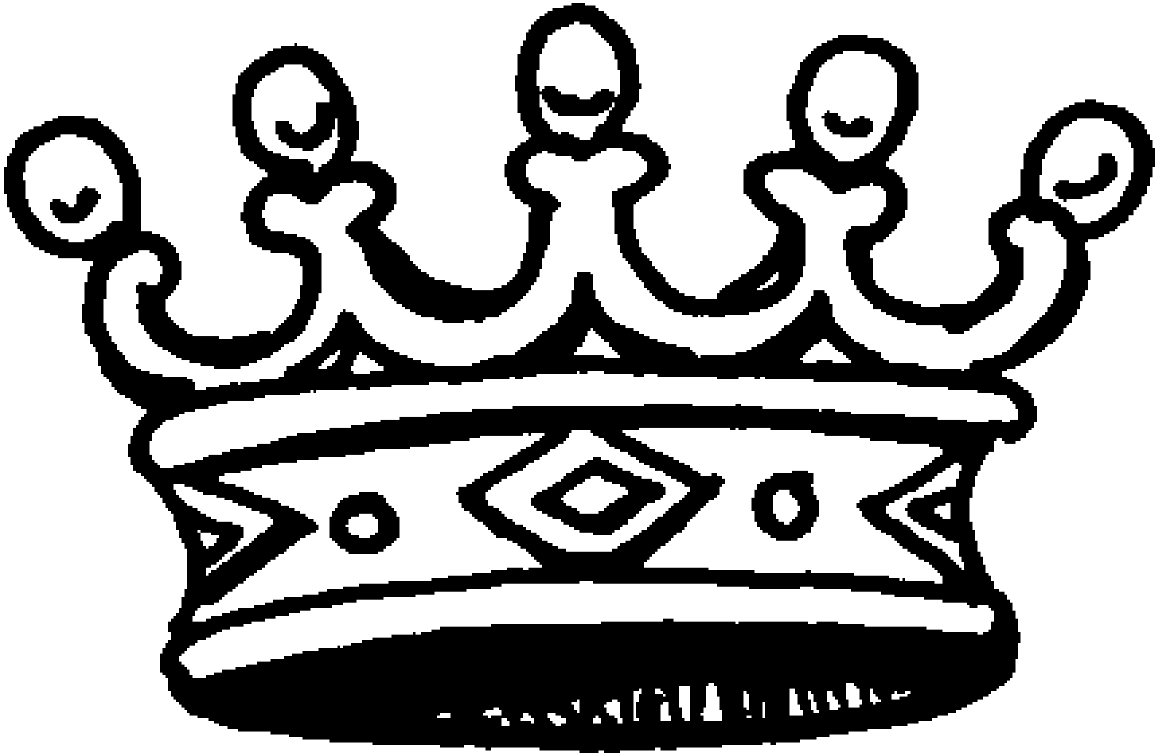
Reduction Rule:
If a candidate line covers a set S of more than k points $\Rightarrow$ $(P \setminus S, k-1)$.

If this rule cannot be applied and there are still more than k^2 points, then there is no solution $\Rightarrow$ Kernel with at most k^2 points.

Kernelization

- Kernelization can be thought of as a polynomial-time preprocessing before attacking the problem with whatever method we have. “It does no harm” to try kernelization.
- Some kernelizations use lots of simple reduction rules and require a complicated analysis to bound the kernel size...
- ... while other kernelizations are based on surprising nice tricks (Next: Crown Reduction and the Sunflower Lemma).
- Possibility to prove lower bounds (S. Saurabh’s lecture).

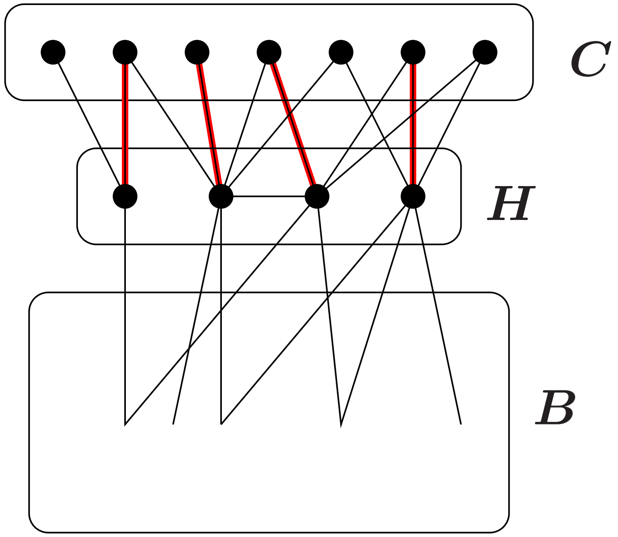
Crown Reduction



Crown Reduction

Definition: A **crown decomposition** is a partition $C \cup H \cup B$ of the vertices such that

- C is an independent set,
- there is no edge between C and B,
- there is a matching between C and H that covers H.



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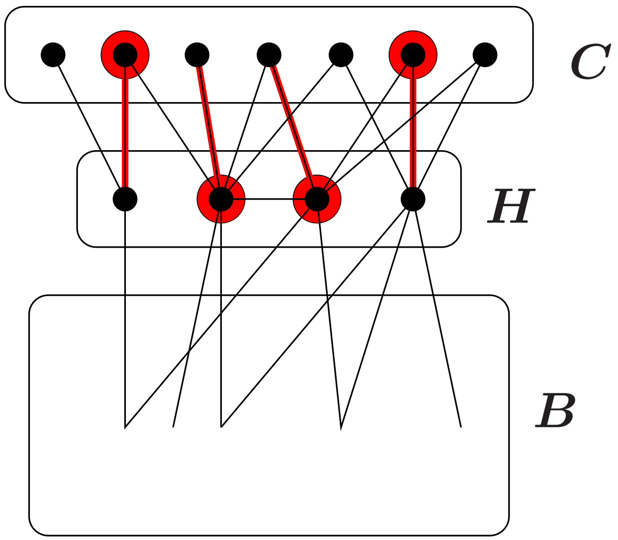

Crown rule for VERTEX COVER:

The matching needs to be covered and we can assume that it is covered by H (makes no sense to use vertices of C)

$\Rightarrow (G \setminus (H \cup C), k - |H|)$.

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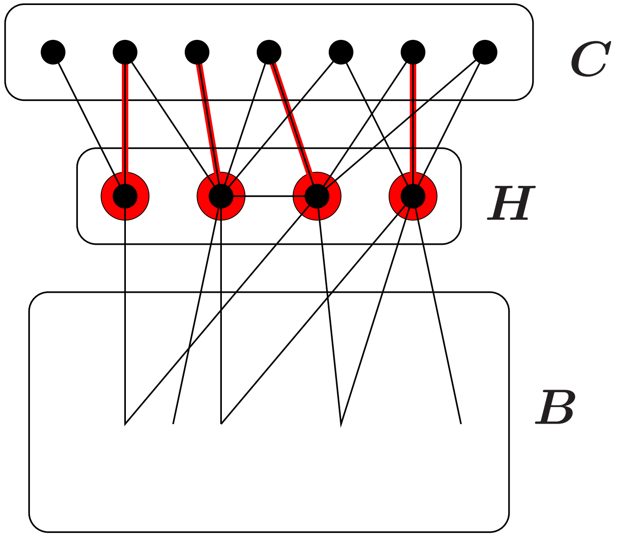

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Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$,
- find a crown decomposition,
- or conclude that the graph has at most $3k$ vertices.

Crown Reduction

Key lemma:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$, $\Rightarrow$ No solution!
- find a crown decomposition, $\Rightarrow$ Reduce!
- or conclude that the graph has at most $3k$ vertices.

  $\Rightarrow$ $3k$ vertex kernel!

This gives a 3k vertex kernel for VERTEX COVER.

Proof

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k + 1$,
- find a crown decomposition,
- or conclude that the graph has at most $3k$ vertices.

For the proof, we need the classical Kőnig's Theorem.

$\tau(G)$: size of the minimum vertex cover
$\nu(G)$: size of the maximum matching (independent set of edges)

Theorem: [Kőnig, 1931] If G is **bipartite,** then

$$\tau(G) = \nu(G)$$

Proof

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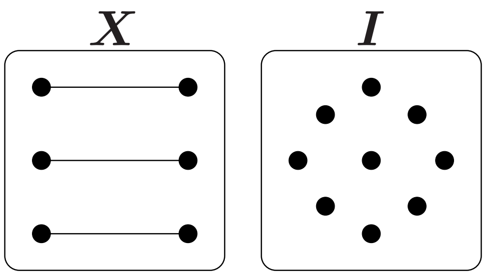
Proof: Find (greedily) a maximal matching; if its size is at least $k + 1$, then we are done. The rest of the graph is an independent set I.



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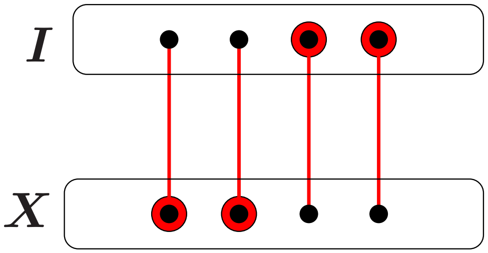

Find a maximum matching/minimum vertex cover in the bipartite graph between X and I.

Proof

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$,
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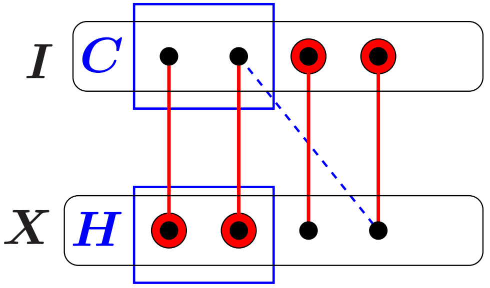
- or conclude that the graph has at most $3k$ vertices.

Proof:

Case 1: The minimum vertex cover contains at least one vertex of X

$\Rightarrow$ There is a crown decomposition.



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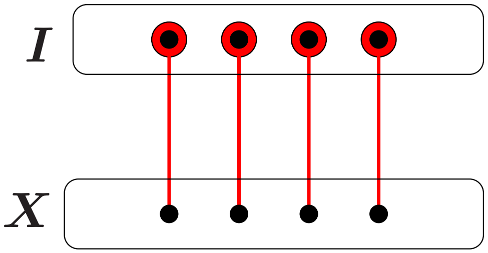

Case 2: The minimum vertex cover contains only vertices of I $\Rightarrow$ It contains every vertex of I
$\Rightarrow$ There are at most $2k+k$ vertices.

DUAL OF VERTEX COLORING

Parameteric dual of k-COLORING. Also known as SAVING k COLORS.

Task: Given a graph G and an integer k, find a vertex coloring with $|V(G)| - k$ colors.

Crown rule for DUAL OF VERTEX COLORING:

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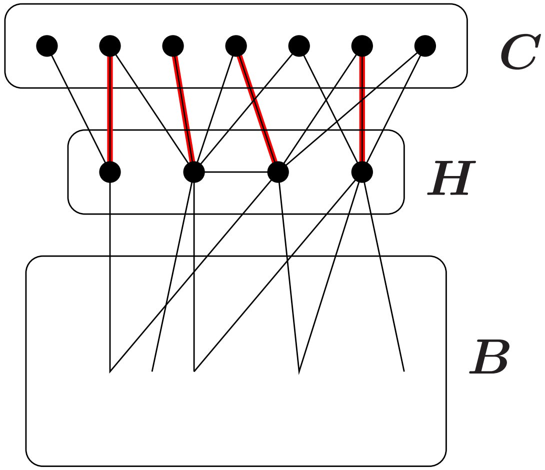
Crown rule for DUAL OF VERTEX COLORING:

Suppose there is a crown decomposition for the **complement graph** $\overline{G}$.

- C is a clique in G: each vertex needs a distinct color.
- Because of the matching, H can be colored using only these $|C|$ colors.
- These colors cannot be used for B.
- $(G \setminus (H \cup C), k - |H|)$



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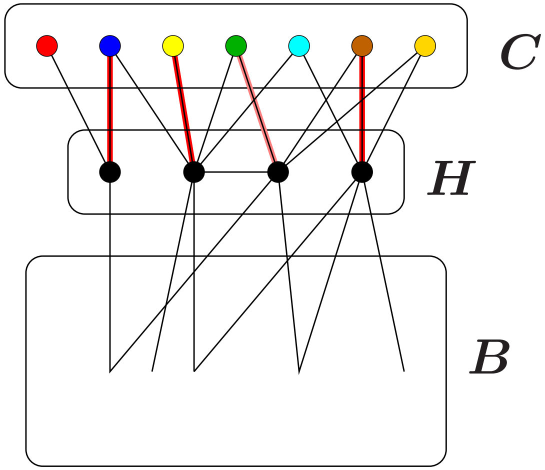

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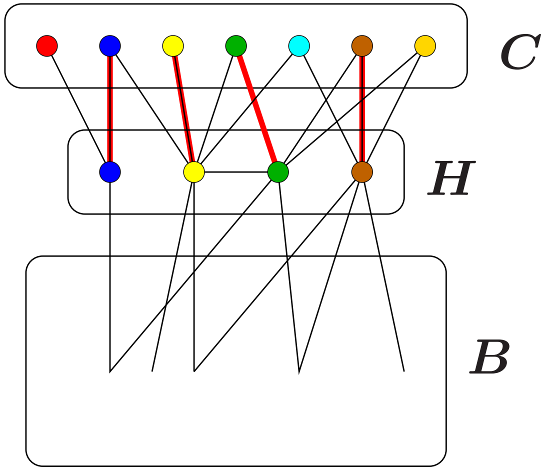


Crown Reduction for **DUAL OF VERTEX COLORING**

Use the key lemma for the complement $\overline{G}$ of G:

Lemma: Given a graph G without isolated vertices and an integer k, in polynomial time we can either

- find a matching of size $k+1$, $\Rightarrow$ YES: we can save k colors!
- find a crown decomposition, $\Rightarrow$ Reduce!
- or conclude that the graph has at most $\mathbf{3}k$ vertices.

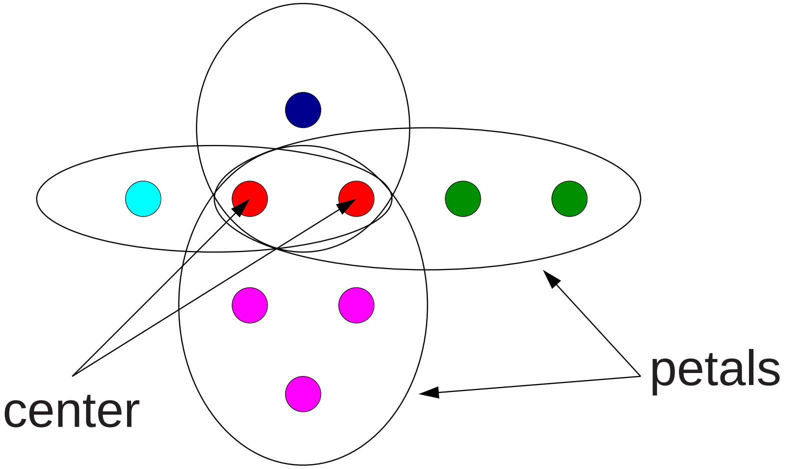
  $\Rightarrow$ $\mathbf{3}k$ vertex kernel!

This gives a 3k vertex kernel for DUAL OF VERTEX COLORING.

Sunflower Lemma

Sunflower lemma

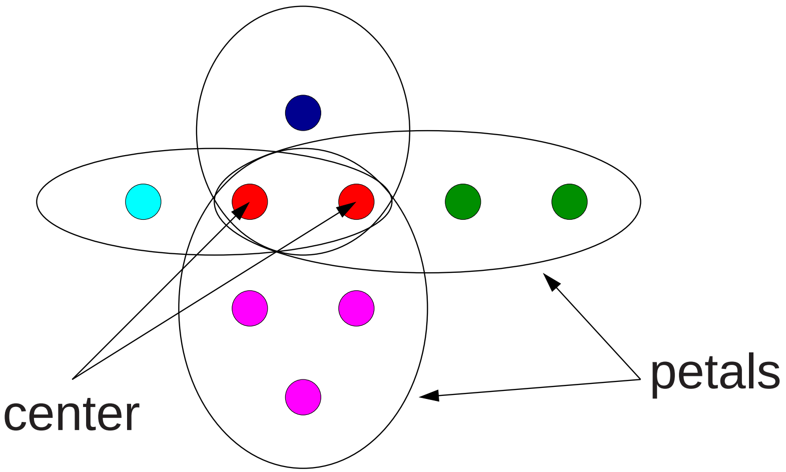
Definition: Sets S_1, S_2, . . . , S_k form a **sunflower** if the sets $S_i \setminus (S_1 \cap S_2 \cap \cdots \cap S_k)$ are disjoint.



Lemma: [Erdős and Rado, 1960] If the size of a set system is greater than $(p-1)^d \cdot d!$ and it contains only sets of size at most d, then the system contains a sunflower with p petals. Furthermore, in this case such a sunflower can be found in polynomial time.

Sunflowers and d**-HITTING SET**

$\boldsymbol{d}$-HITTING SET: Given a collection $\boldsymbol{\mathcal{S}}$ of sets of size at most $\boldsymbol{d}$ and an integer $\boldsymbol{k}$, find a set $\boldsymbol{S}$ of $\boldsymbol{k}$ elements that intersects every set of $\boldsymbol{\mathcal{S}}$.

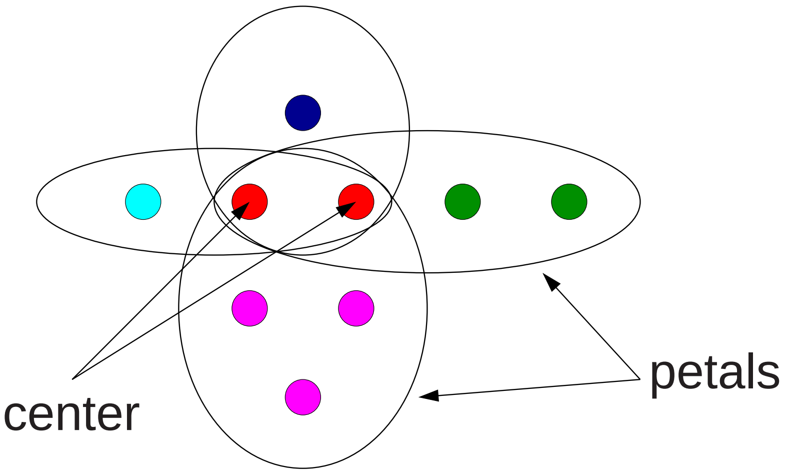


Reduction Rule: If $\boldsymbol{k+1}$ sets form a sunflower, then remove these sets from $\boldsymbol{\mathcal{S}}$ and add the center $\boldsymbol{C}$ to $\boldsymbol{\mathcal{S}}$ ($\boldsymbol{S}$ does not hit one of the petals, thus it has to hit the center).

If the rule cannot be applied, then there are at most $\boldsymbol{O(k^d)}$ sets.

Sunflowers and d-HITTING SET

d-HITTING SET: Given a collection $\mathcal{S}$ of sets of size at most d and an integer k, find a set S of k elements that intersects every set of $\mathcal{S}$.



Reduction Rule (variant): Suppose more than $k+1$ sets form a sunflower.

- If the sets are disjoint $\Rightarrow$ No solution.
- Otherwise, keep only $k+1$ of the sets.

If the rule cannot be applied, then there are at most $O(k^d)$ sets.

Graph Minors

Neil Robertson

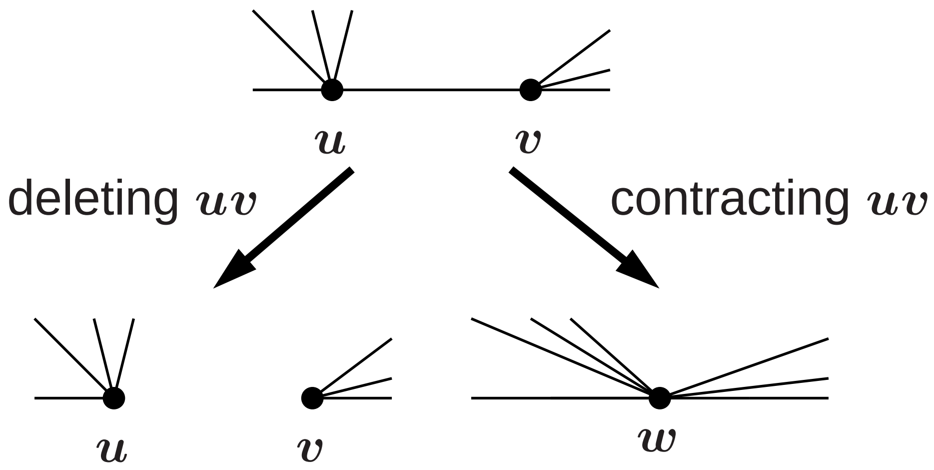
Paul Seymour

Graph Minors

- Some consequences of the Graph Minors Theorem give a quick way of showing that certain problems are FPT.
- However, the function $f(k)$ in the resulting FPT algorithms can be HUGE, completely impractical.
- History: motivation for FPT.
- Parts and ingredients of the theory are useful for algorithm design.
- New algorithmic results are still being developed.

Graph Minors

Definition: Graph H is a **minor** G ($H \leq G$) if H can be obtained from G by deleting edges, deleting vertices, and contracting edges.

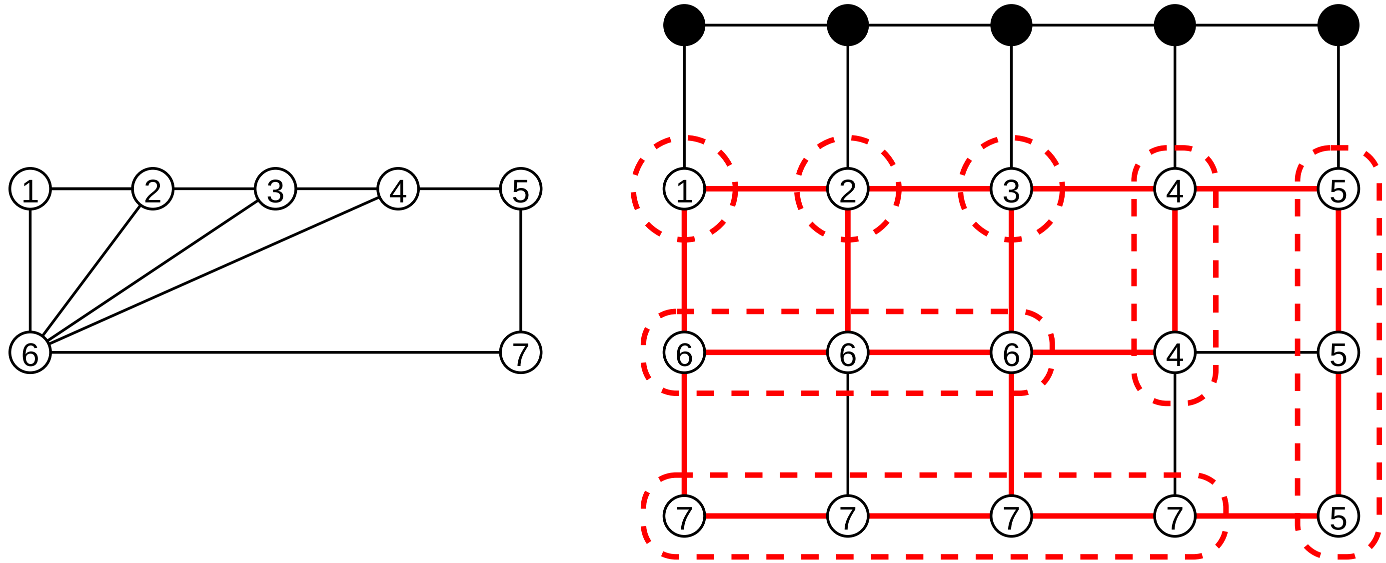


Example: A triangle is a minor of a graph G if and only if G has a cycle (i.e., it is not a forest).

Graph minors

Equivalent definition: Graph H is a **minor** of G if there is a mapping ϕ that maps each vertex of H to a connected subset of G such that

- $\phi(u)$ and $\phi(v)$ are disjoint if $u \neq v$, and
- if $uv \in E(G)$, then there is an edge between $\phi(u)$ and $\phi(v)$.



Minor closed properties

Definition: A set $\mathcal{G}$ of graphs is **minor closed** if whenever $G \in \mathcal{G}$ and $H \leq G$, then $H \in \mathcal{G}$ as well.

Examples of minor closed properties:

planar graphs

acyclic graphs (forests)

graphs having no cycle longer than k

empty graphs

Examples of not minor closed properties:

complete graphs

regular graphs

bipartite graphs

Forbidden minors

Let $\mathcal{G}$ be a minor closed set and let $\mathcal{F}$ be the set of "minimal bad graphs": $H \in \mathcal{F}$ if $H \notin \mathcal{G}$, but every proper minor of H is in $\mathcal{G}$.

Characterization by forbidden minors:

$$G \in \mathcal{G} \iff \forall H \in \mathcal{F}, H \not\leq G$$

The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

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The set $\mathcal{F}$ is the **obstruction set** of property $\mathcal{G}$.

Theorem: [Wagner] A graph is planar if and only if it does not have a K_5 or $K_{3,3}$ minor.

In other words: the obstruction set of planarity is $\mathcal{F} = \{K_5, K_{3,3}\}$.

Does every minor closed property have such a finite characterization?

Graph Minors Theorem

Theorem: [Robertson and Seymour] Every minor closed property $\mathcal{G}$ has a finite obstruction set.

Note: The proof is contained in the paper series “Graph Minors I–XX”.
Note: The size of the obstruction set can be astronomical even for simple properties.

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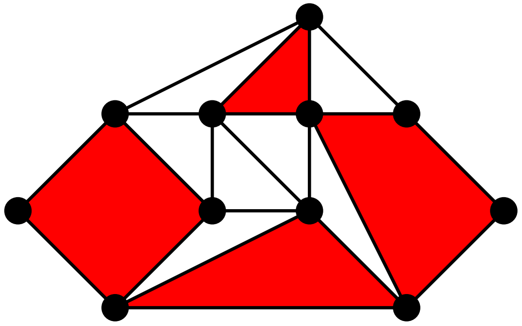
Note: The size of the obstruction set can be astronomical even for simple properties.

Theorem: [Robertson and Seymour] For every fixed graph H, there is an $O(n^3)$ time algorithm for testing whether H is a minor of the given graph G.

Corollary: For every minor closed property $\mathcal{G}$, there is an $O(n^3)$ time algorithm for testing whether a given graph G is in $\mathcal{G}$.

Applications

PLANAR FACE COVER: Given a graph G and an integer k, find an embedding of planar graph G such that there are k faces that cover all the vertices.



One line argument:

For every fixed k, the class $\mathcal{G}_k$ of graphs of yes-instances is minor closed.

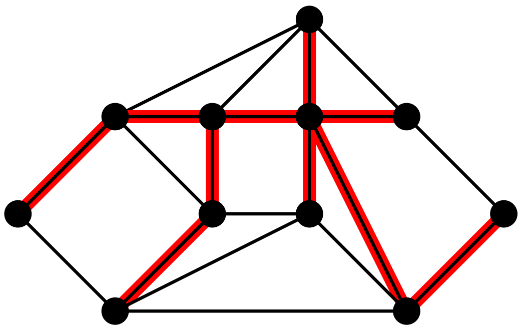
$$\Downarrow$$

For every fixed k, there is a $O(n^3)$ time algorithm for PLANAR FACE COVER.

Note: non-uniform FPT.

Applications

$\boldsymbol{k}$-LEAF SPANNING TREE: Given a graph $\boldsymbol{G}$ and an integer $\boldsymbol{k}$, find a spanning tree with **at least** $\boldsymbol{k}$ leaves.



Technical modification: Is there such a spanning tree for at least one component of $\boldsymbol{G}$?

One line argument:

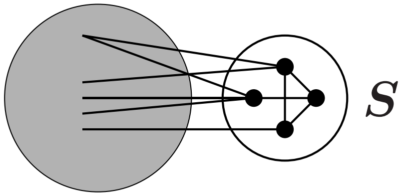
For every fixed $\boldsymbol{k}$, the class $\boldsymbol{\mathcal{G}_k}$ of no-instances is minor closed.

$$\Downarrow$$

For every fixed $\boldsymbol{k}$, $\boldsymbol{k}$-LEAF SPANNING TREE can be solved in time $\boldsymbol{O(n^3)}$.

$\mathcal{G} + k$ vertices

Let $\mathcal{G}$ be a graph property, and let $\mathcal{G} + kv$ contain graph G if there is a set $S \subseteq V(G)$ of k vertices such that $G \setminus S \in \mathcal{G}$.



Lemma: If $\mathcal{G}$ is minor closed, then $\mathcal{G} + kv$ is minor closed for every fixed k.
$\Rightarrow$ Finding the smallest k such that a given graph is in $\mathcal{G} + kv$ is FPT.

$\mathcal{G} + k$ vertices

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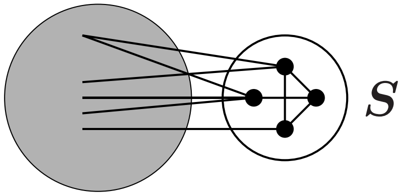

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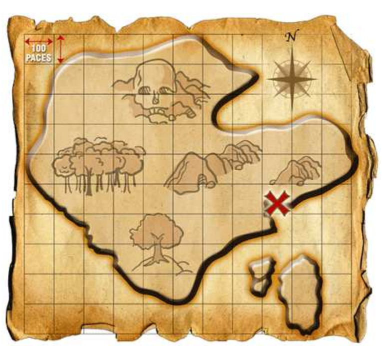
$\Rightarrow$ Finding the smallest k such that a given graph is in $\mathcal{G} + kv$ is FPT.

- If $\mathcal{G} =$ forests $\Rightarrow \mathcal{G} + kv =$ graphs that can be made acyclic by the deletion of k vertices $\Rightarrow$ FEEDBACK VERTEX SET is FPT.
- If $\mathcal{G} =$ planar graphs $\Rightarrow \mathcal{G} + kv =$ graphs that can be made planar by the deletion of k vertices (k-apex graphs) $\Rightarrow$ k-APEX GRAPH is FPT.
- If $\mathcal{G} =$ empty graphs $\Rightarrow \mathcal{G} + kv =$ graphs with vertex cover number at most $k \Rightarrow$ VERTEX COVER is FPT.

Two types of problems

We have to solve some problems.

We have to find something nice hidden somewhere.



Two types of problems

We have to solve some problems.

Typically **minimization** problems: Vertex Cover, Hitting Set, Dominating Set, covering/stabbing problems, graph modification problems, ...

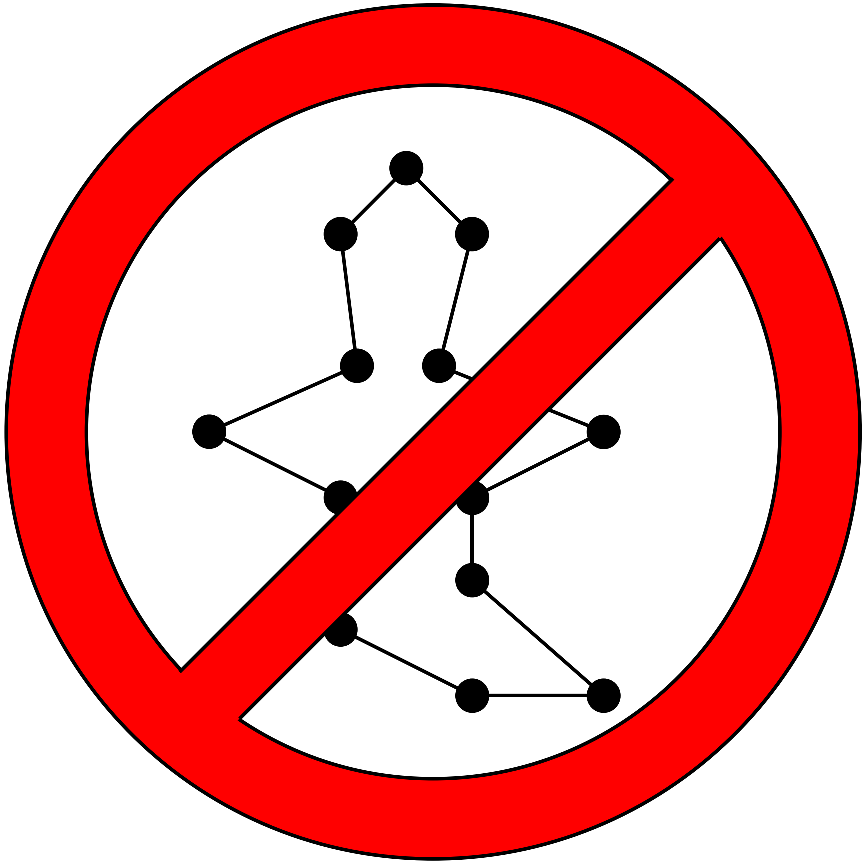
Bounded search trees, iterative compression

We have to find something nice hidden somewhere.

Typically **maximization** problems: k-Path, Disjoint Triangles, k-Leaf Spanning Tree, ...

Color coding, matroids

Forbidden subgraphs



Forbidden subgraphs

General problem class: Given a graph G and an integer k, transform G with at most k modifications (add/remove vertices/edges) into a graph having property $\mathcal{P}$.

Example:

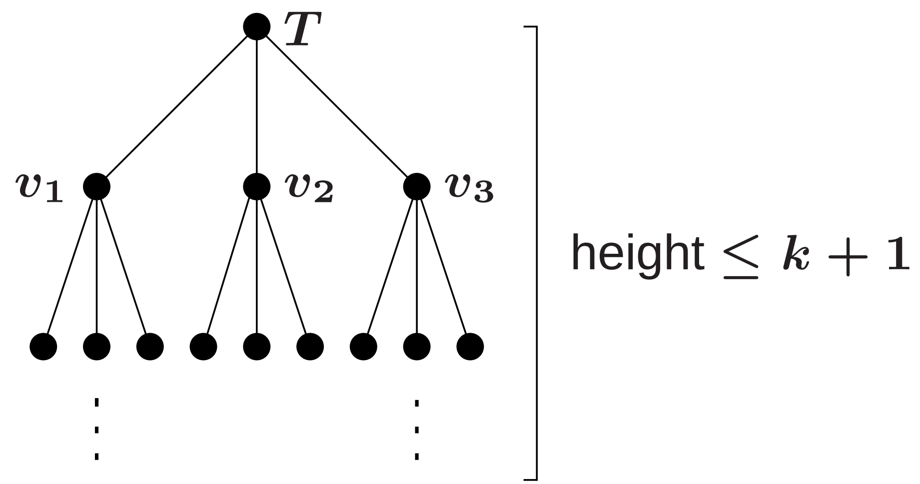
TRIANGLE DELETION: make the graph triangle-free by deleting at most k vertices.

Branching algorithm:

- If the graph is triangle-free, then we are done.
- If there is a triangle $v_1v_2v_3$, then at least one of v_1, v_2, v_3 has to be deleted $\Rightarrow$ We branch into 3 directions.

TRIANGLE DELETION

Search tree:



The search tree has at most 3^k leaves and the work to be done is polynomial at each step $\Rightarrow O^*(3^k)$ time algorithm.

Note: If the answer is "NO", then the search tree has **exactly** 3^k leaves.

Hereditary properties

Definition: A graph property $\mathcal{P}$ is **hereditary** if for every $G \in \mathcal{P}$ and induced subgraph G' of G, we have $G' \in \mathcal{P}$ as well.

Examples: triangle-free, bipartite, interval graph, planar

Observation: Every hereditary property $\mathcal{P}$ can be characterized by a (finite or infinite) set $\mathcal{F}$ of forbidden induced subgraphs:

$$G \in \mathcal{P} \Leftrightarrow \forall H \in \mathcal{F}, H \not\subseteq_{\text{ind}} G$$

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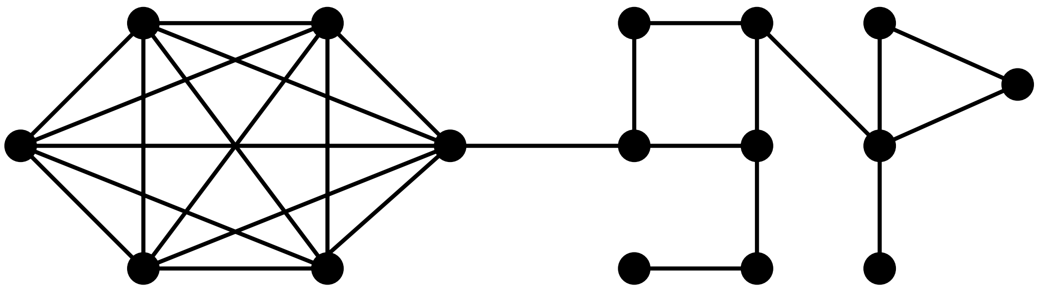
Proof:

- Suppose that every graph in $\mathcal{F}$ has at most r vertices. Using brute force, we can find in time $O(n^r)$ a forbidden subgraph (if exists).
- If a forbidden subgraph exists, then we have to delete one of the at most r vertices or add/delete one of the at most $\binom{r}{2}$ edges $\Rightarrow$ Branching factor is a constant c depending on $\mathcal{F}$.
- The search tree has at most c^k leaves and the work to be done at each node is $O(n^r)$.

Cluster Editing

Task: Given a graph G and an integer k, add/remove at most k edges such that every component is a clique in the resulting graph.



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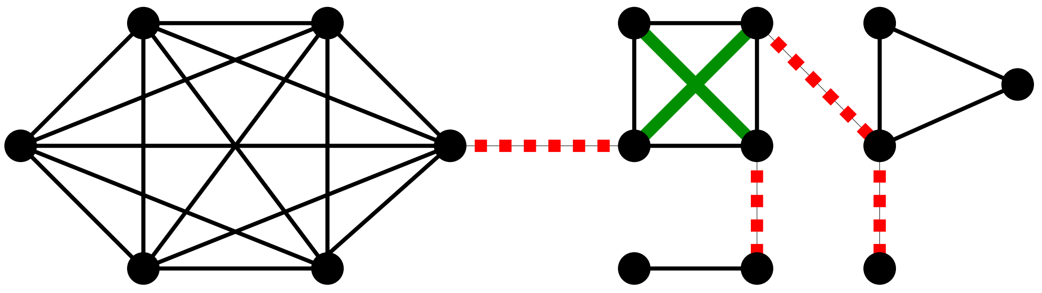

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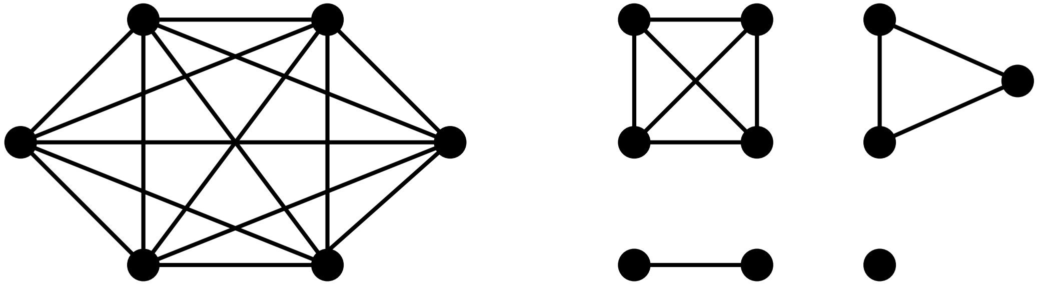

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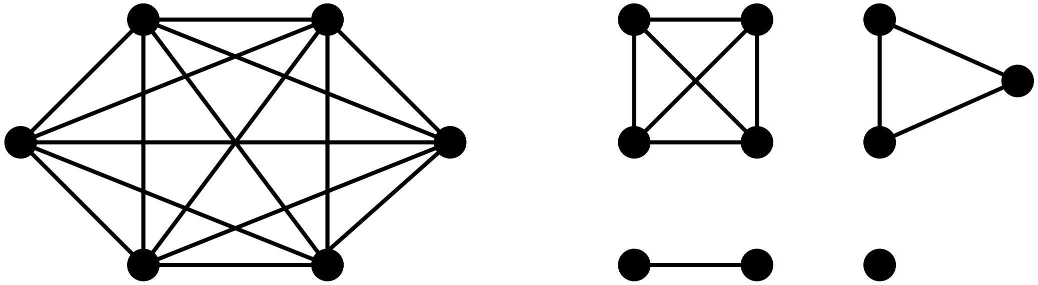
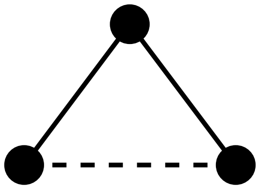


Property $\mathcal{P}$: every component is a clique.

Forbidden induced subgraph:



$O^*(3^k)$ time algorithm.

CHORDAL COMPLETION

Definition: A graph is **chordal** if it does not contain an induced cycle of length greater than 3.

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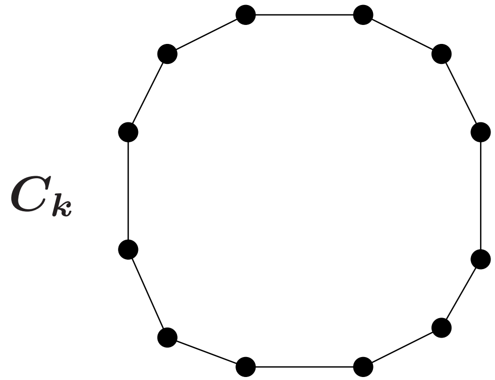
Proof: By induction. $k=3$ is trivial.



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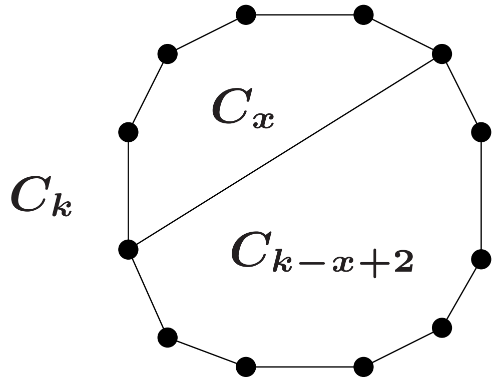


C_x: $x-3$ edges
C_{k-x+2}: $k-x-1$ edges
C_k: $(x-3)+(k-x-1)+1 =$
$k-3$ edges

CHORDAL COMPLETION

Algorithm:

- Find an induced cycle C of length at least 4 (can be done in polynomial time).
- If no such cycle exists $\Rightarrow$ Done!
- If C has more than $k+3$ vertices $\Rightarrow$ No solution!
- Otherwise, one of the

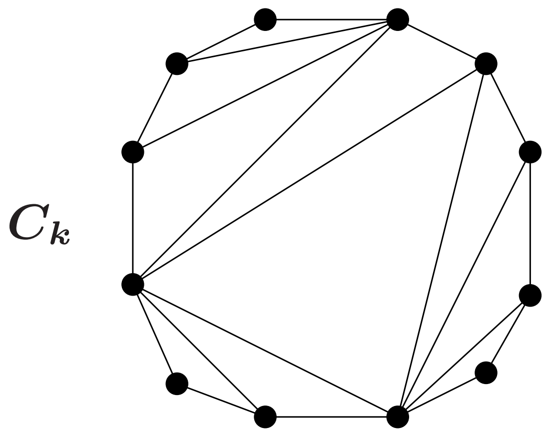
$$\binom{|C|}{2} - |C| \leq (k+3)(k+2)/2 - k = O(k^2)$$

missing edges has to be added $\Rightarrow$ Branch!

Size of the search tree is $k^{O(k)}$.

CHORDAL COMPLETION – *more efficiently*

Definition: Triangulation of a cycle.



Lemma: Every chordal supergraph of a cycle C contains a triangulation of the cycle C.

Lemma: The number of ways a cycle of length k can be triangulated is exactly the $(k-2)$th Catalan number

$$C_{k-2} = \frac{1}{k-1}\binom{2(k-2)}{k-2} \leq 4^{k-3}.$$

CHORDAL COMPLETION – *more efficiently*

Algorithm:

- Find an induced cycle C of length at least 4 (can be done in polynomial time).
- If no such cycle exists $\Rightarrow$ Done!
- If C has more than $k + 3$ vertices $\Rightarrow$ No solution!
- Otherwise, one of the $\leq 4^{|C|-3}$ triangulations has to be in the solution $\Rightarrow$ Branch!

Claim: Search tree has at most $T_k = 4^k$ leaves.

Proof: By induction. Number of leaves is at most

$$T_k \leq 4^{|C|-3} \cdot T_{k-(|C|-3)} \leq 4^{|C|-3} \cdot 4^{k-(|C|-3)} = 4^k.$$

Iterative compression

Iterative compression

- A surprising small, but very powerful trick.
- Most useful for deletion problems: delete k things to achieve some property.
- Demonstration: ODD CYCLE TRANSVERSAL aka BIPARTITE DELETION aka GRAPH BIPARTIZATION: Given a graph G and an integer k, delete k vertices to make the graph bipartite.
- Forbidden induced subgraphs: odd cycles. There is no bound on the size of odd cycles.

BIPARTITE DELETION

Solution based on iterative compression:

- **Step 1:**

  Solve the **annotated problem** for bipartite graphs:

  Given a bipartite graph G, two sets $B, W \subseteq V(G)$, and an integer k, find a set S of at most k vertices such that $G \setminus S$ has a 2-coloring where $B \setminus S$ is black and $W \setminus S$ is white.

Bipartite Deletion

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- **Step 2:**

  Solve the **compression problem** for general graphs:

  Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.

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- **Step 3:**

  Apply the magic of iterative compression$\ldots$

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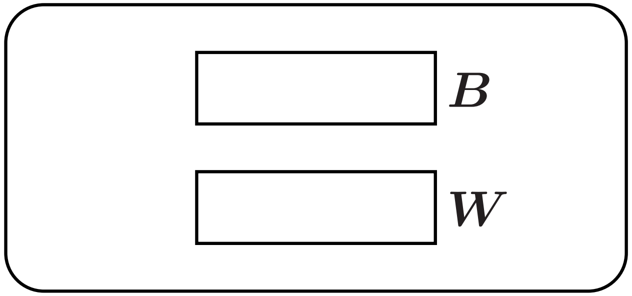

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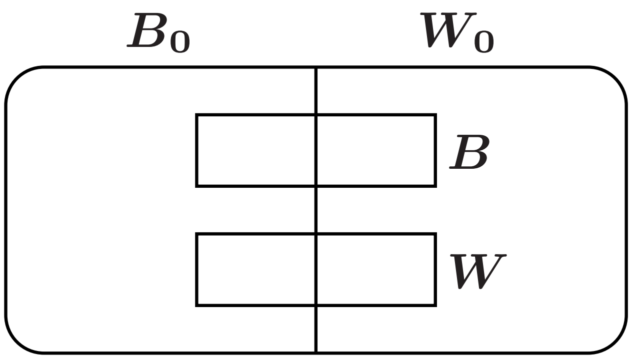

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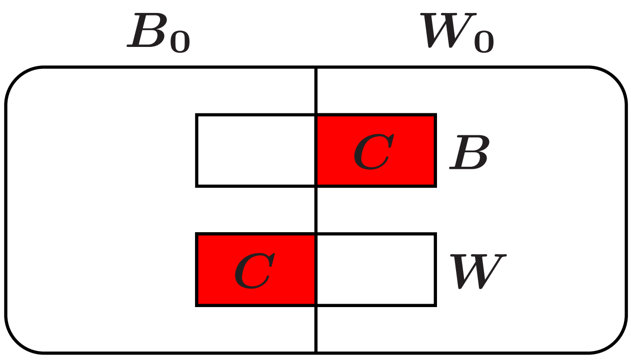

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$C := (B_0 \cap W) \cup (W_0 \cap B)$ should change color, while
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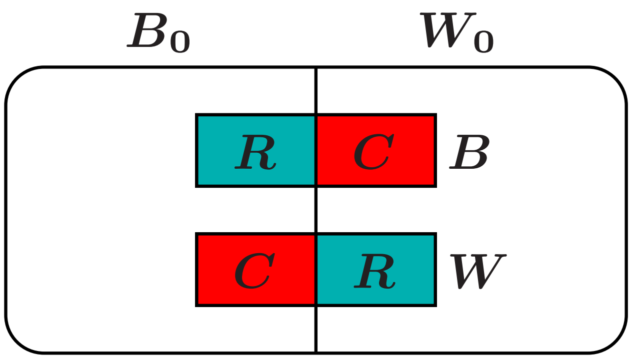

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Lemma: $G \setminus S$ has the required 2-coloring if and only if S separates C and R, i.e., no component of $G \setminus S$ contains vertices from both $C \setminus S$ and $R \setminus S$.

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Proof:

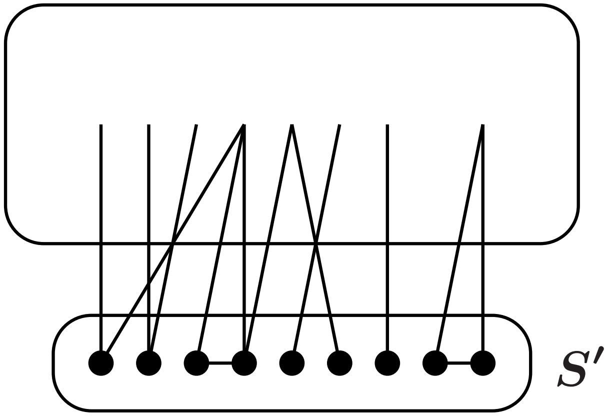
$\Rightarrow$ In a 2-coloring of $G \setminus S$, each vertex either remained the same color or changed color. Adjacent vertices do the same, thus every component either changed or remained.

$\Leftarrow$ Flip the coloring of those components of $G \setminus S$ that contain vertices from $C \setminus S$. No vertex of R is flipped.

Algorithm: Using max-flow min-cut techniques, we can check if there is a set S that separates C and R. It can be done in time $O(k|E(G)|)$ using k iterations of the Ford-Fulkerson algorithm.

Step 2: The compression problem

Given a graph G, an integer k, and a set S' of $k+1$ vertices such that $G \setminus S'$ is bipartite, find a set S of k vertices such that $G \setminus S$ is bipartite.



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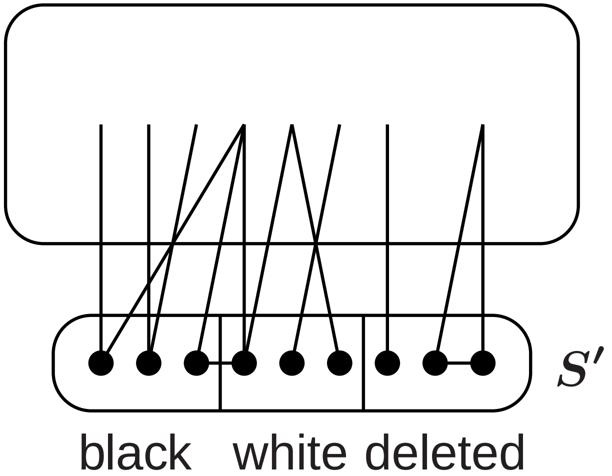

Branch into 3^{k+1} cases: each vertex of S' is either black, white, or deleted.

Trivial check: no edge between two black or two white vertices.

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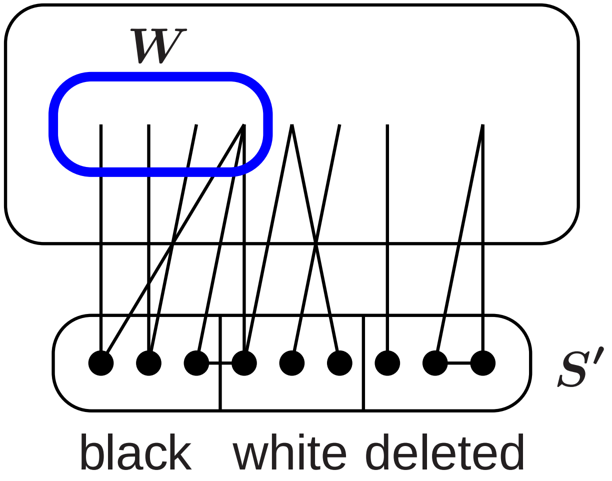

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Neighbors of the black vertices in S' should be white and the neighbors of the white vertices in S' should be black.

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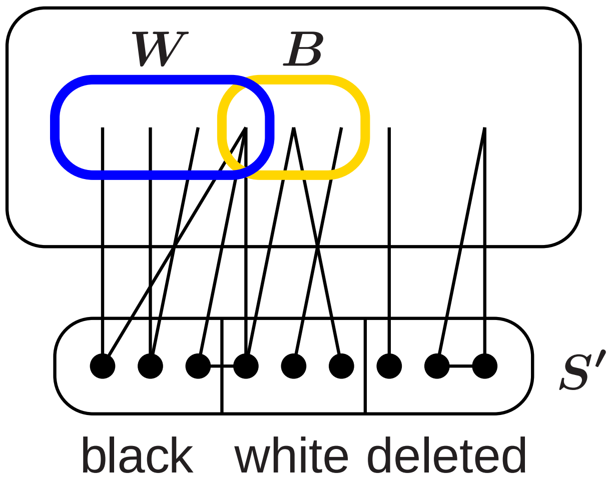

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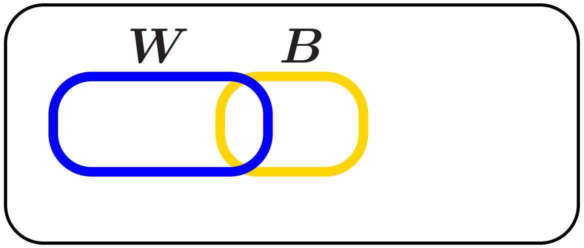

The vertices of S' can be disregarded. Thus we need to solve the annotated problem on the bipartite graph $G \setminus S'$.

Running time: $O(3^k \cdot k|E(G)|)$ time.

Step 3: Iterative compression

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Let $V(G) = \{v_1, \ldots, v_n\}$ and let G_i be the graph induced by $\{v_1, \ldots, v_i\}$.

For every i, we find a set S_i of size k such that $G_i \setminus S_i$ is bipartite.

- For G_k, the set $S_k = \{v_1, \ldots, v_k\}$ is a trivial solution.
- If S_{i-1} is known, then $S_{i-1} \cup \{v_i\}$ is a set of size $k+1$ whose deletion makes G_i bipartite $\Rightarrow$ We can use the compression algorithm to find a suitable S_i in time $O(3^k \cdot k|E(G_i)|)$.

Step 3: Iterative Compression

Bipartite-Deletion(G, k)

1. $S_k = \{v_1, \ldots, v_k\}$
2. for $i := k + 1$ to n
3. Invariant: $G_{i-1} \setminus S_{i-1}$ is bipartite.
4. Call Compression$(G_i, S_{i-1} \cup \{v_i\})$
5. If the answer is “NO” $\Rightarrow$ return “NO”
6. If the answer is a set $X \Rightarrow S_i := X$
7. Return the set S_n

Running time: the compression algorithm is called n times and everything else can be done in linear time

$\Rightarrow O(3^k \cdot k|V(G)| \cdot |E(G)|)$ time algorithm.

Color coding

Color coding

- Works best when we need to ensure that a small number of “things” are disjoint.
- We demonstrate it on two problems:
  - Find an s-t path of length exactly k.
  - Find k vertex-disjoint triangles in a graph.
- Randomized algorithm, but can be derandomized using a standard technique.
- Very robust technique, we can use it as an “opening step” when investigating a new problem.

k-PATH

Task: Given a graph G, an integer k, two vertices s, t, find a **simple** s-t path with exactly k internal vertices.

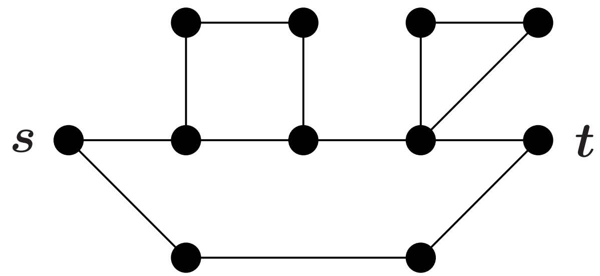
Note: Finding such a **walk** can be done easily in polynomial time.

Note: The problem is clearly NP-hard, as it contains the s-t HAMILTONIAN PATH problem.

The k-PATH algorithm can be used to check if there is a cycle of length exactly k in the graph.

k-PATH

- Assign colors from $[k]$ to vertices $V(G) \setminus \{s, t\}$ uniformly and independently at random.



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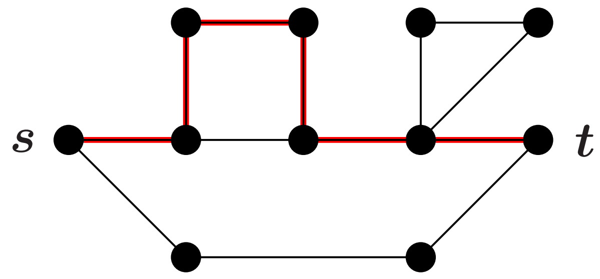

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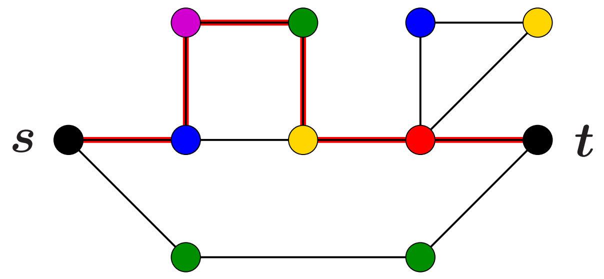

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output “YES” or “NO”.

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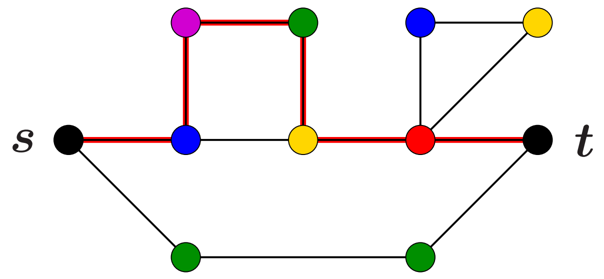

- Check if there is a **colorful** s-t path: a path where each color appears exactly once on the internal vertices; output "YES" or "NO".
  - If there is no s-t k-path: no such colorful path exists $\Rightarrow$ "NO".
  - If there is an s-t k-path: the probability that such a path is colorful is

$$\frac{k!}{k^k} > \frac{\left(\frac{k}{e}\right)^k}{k^k} = e^{-k},$$

thus the algorithm outputs "YES" with at least that probability.

Error probability

- If there is a k-path, the probability that the algorithm **does not** say "YES" after e^k repetitions is at most

$$(1 - e^{-k})^{e^k} < \left(e^{-e^{-k}}\right)^{e^k} = 1/e \approx 0.38$$

- Repeating the whole algorithm a constant number of times can make the error probability an arbitrary small constant.
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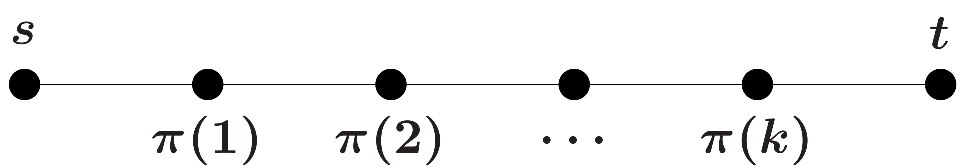
It remains to see how a colorful s-t path can be found.

Method 1: Trying all permutations.
Method 2: Dynamic programming.

Method 1: Trying all permutations

The colors encountered on a colorful s-t path form a permutation π of $\{1, 2, \ldots, k\}$:



We try all possible $k!$ permutations. For a fixed π, it is easy to check if there is a path with this order of colors.

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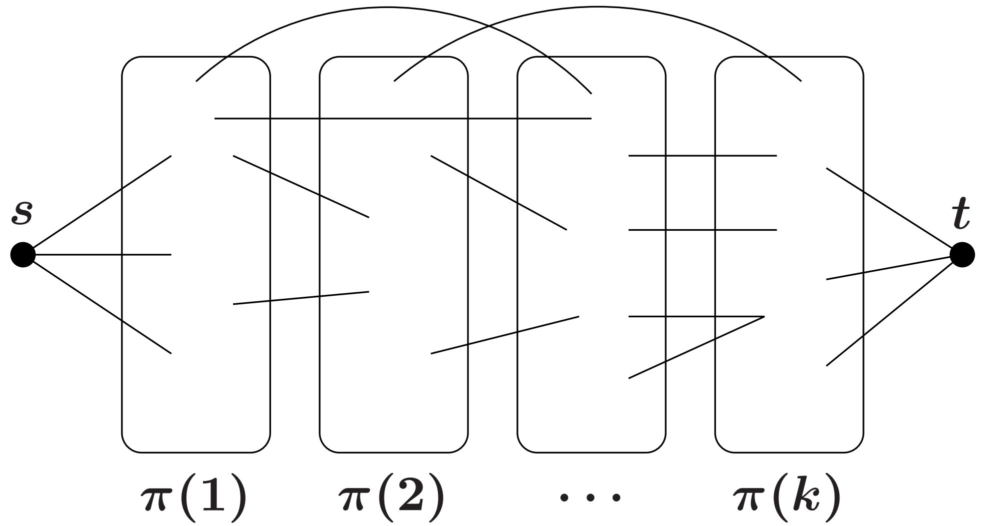

- Edges connecting nonadjacent color classes are removed.
- The remaining edges are directed.
- All we need to check if there is a directed s-t path.
- Running time is $O(k! \cdot |E(G)|)$.

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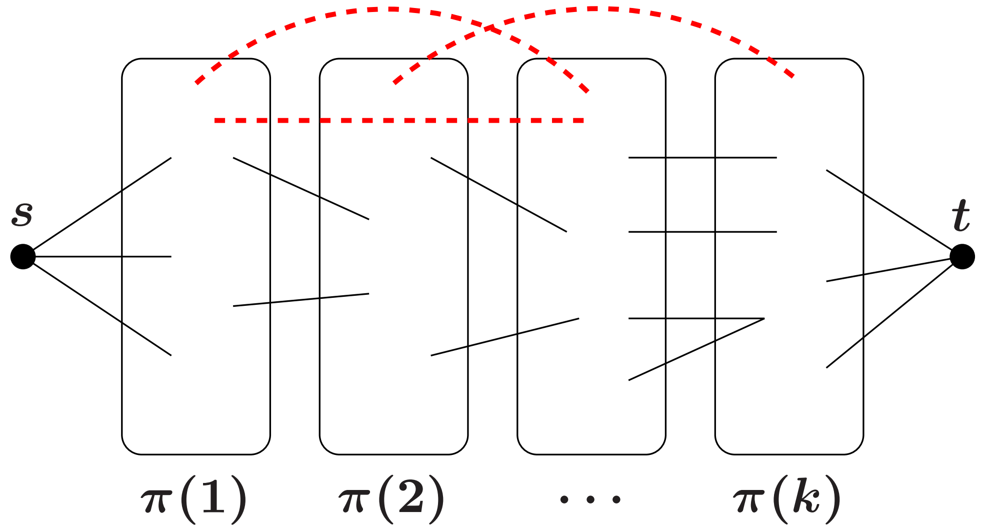

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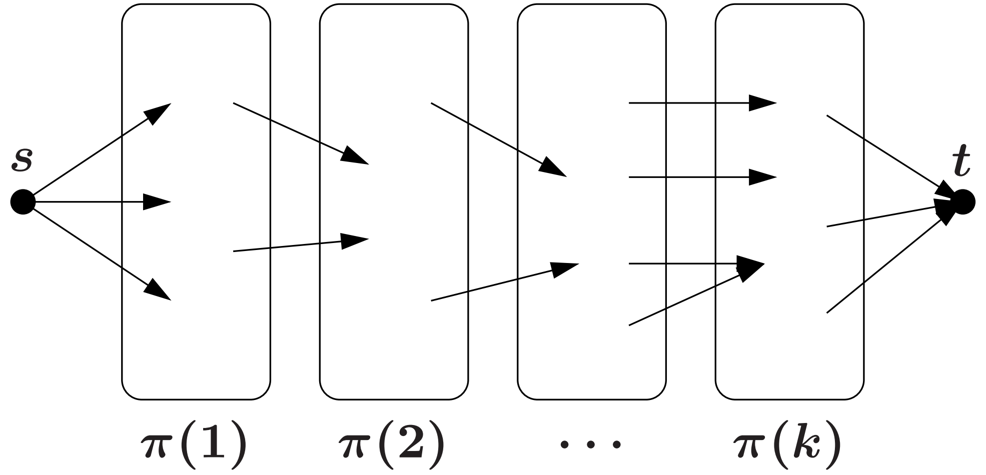

- Edges connecting nonadjacent color classes are removed.
- The remaining edges are directed.
- All we need to check if there is a directed s-t path.
- Running time is $O(k! \cdot |E(G)|)$.

Method 2: Dynamic Programming

We introduce $2^k \cdot |V(G)|$ Boolean variables:

$x(v, C) =$ TRUE for some $v \in V(G)$ and $C \subseteq [k]$

$$\Updownarrow$$

There is an s-v path where each color in C appears exactly once and no other color appears.

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If we know every $x(v, C)$ with $|C| = i$, then we can determine every $x(v, C)$ with $|C| = i + 1 \Rightarrow$ All the values can be determined in time $O(2^k \cdot |E(G)|)$.

There is a colorful s-t path $\Leftrightarrow x(v, [k]) =$ TRUE for some neighbor of t.

Derandomization

Using Method 2, we obtain a $O^*((2e)^k)$ time algorithm with constant error probability. How to make it deterministic?

Definition: A family $\mathcal{H}$ of functions $[n] \rightarrow [k]$ is a k**-perfect** family of hash functions if for every $S \subseteq [n]$ with $|S| = k$, there is a $h \in \mathcal{H}$ such that $h(x) \neq h(y)$ for any $x, y \in S$, $x \neq y$.

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Instead of trying $O(e^k)$ random colorings, we go through a k-perfect family $\mathcal{H}$ of functions $V(G) \longrightarrow [k]$. If there is a solution $\Rightarrow$ The internal vertices S are colorful for at least one $h \in \mathcal{H}$ $\Rightarrow$ Algorithm outputs “YES”.

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Theorem: There is a k-perfect family of functions $[n] \longrightarrow [k]$ having size $2^{O(k)} \log n$.

$\Rightarrow$ There is a **deterministic** $2^{O(k)} \cdot n^{O(1)}$ time algorithm for the k-PATH problem.

k-DISJOINT TRIANGLES

Task: Given a graph G and an integer k, find k vertex disjoint triangles.

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Step 2: Check if there is a colorful solution, where the $3k$ vertices of the k triangles use distinct colors.

- **Method 1:** Try every permutation π of $[3k]$ and check if there are triangles with colors $(\pi(1), \pi(2), \pi(3))$, $(\pi(4), \pi(5), \pi(6))$, $\ldots$
- **Method 2:** Dynamic programming. For $C \subseteq [3k]$ and $|C| = 3i$, let $x(C) =$ TRUE if and only if there are $|C|/3$ disjoint triangles using exactly the colors in C.

$$x(C) = \bigvee_{\{c_1,c_2,c_3\} \subseteq C} (x(C \setminus \{c_1, c_2, c_3\}) \wedge \exists\triangle \text{ with colors } c_1, c_2, c_3)$$

k-DISJOINT TRIANGLES

Step 3: Colorful solution exists with probability at least e^{-3k}, which is a lower bound on the probability of a correct answer.

Running time: constant error probability after e^{3k} repetitions $\Rightarrow$ running time is $O^*((2e)^{3k})$ (using Method 2).

Derandomization: $3k$-perfect family of functions instead of random coloring. Running time is $2^{O(k)} \cdot n^{O(1)}$.

Color coding

We have seen that color coding can be used to find paths, cycles of length k, or a set of k disjoint triangles.

What other structures can be found efficiently with this technique?

The key is treewidth:

Theorem: Given two graph H, G, it can be decided if H is a subgraph of G in time $2^{O(|V(H)|)} \cdot |V(G)|^{O(w)}$, where w is the treewidth of G.

Thus if H belongs to a class of graphs with bounded treewidth, then the subgraph problem is FPT.

Matroid Theory

Matroid Theory

- Matroids: a classical subject of combinatorial optimization.
- Matroids lurk behind matching, flow, spanning tree, and some linear algebra problems.
- A general FPT result that can be used to show that some concrete problems are FPT.

Matroids

Definition: A set system $\mathcal{M}$ over E is a **matroid** if

(1) $\emptyset \in \mathcal{M}$.

(2) If $X \in \mathcal{M}$ and $Y \subseteq X$, then $Y \in \mathcal{M}$.

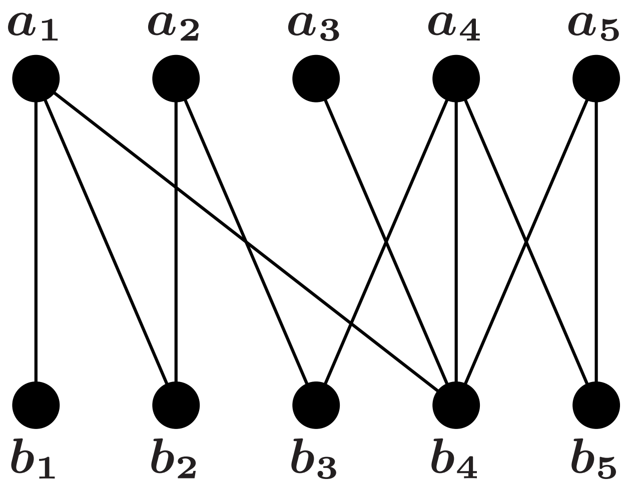
(3) If $X, Y \in \mathcal{M}$ and $|X| > |Y|$, then $\exists e \in X \setminus Y$ such that $Y \cup \{e\} \in \mathcal{M}$.

Example: $\mathcal{M} = \{\emptyset, 1, 2, 3, 12, 13\}$ is a matroid.
Example: $\mathcal{M} = \{\emptyset, 1, 2, 12, 3\}$ is not a matroid.

If $X \in \mathcal{M}$, then we say that X is **independent** in matroid $\mathcal{M}$.

Transversal matroid

Fact: Let $G(A, B; E)$ be a bipartite graph. Those subsets of A that can be covered by a matching form a matroid.



(1) The empty set can be clearly covered.

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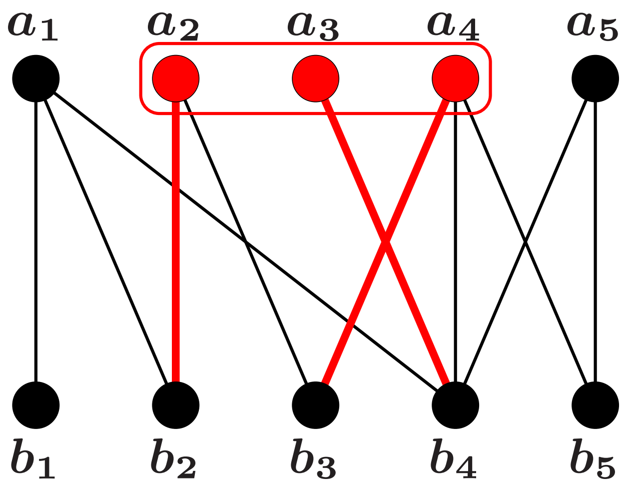

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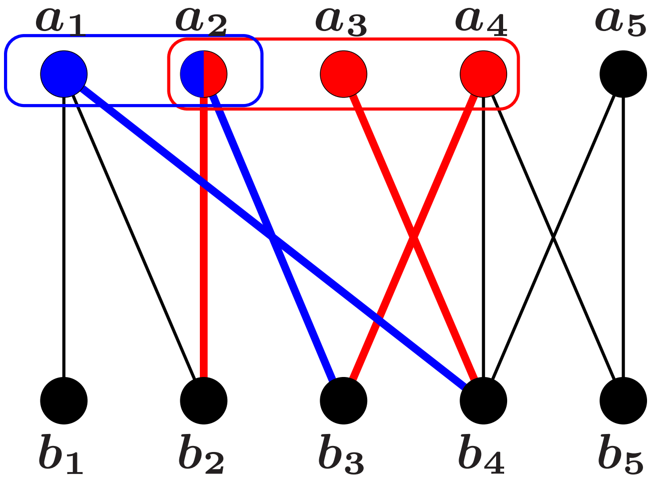

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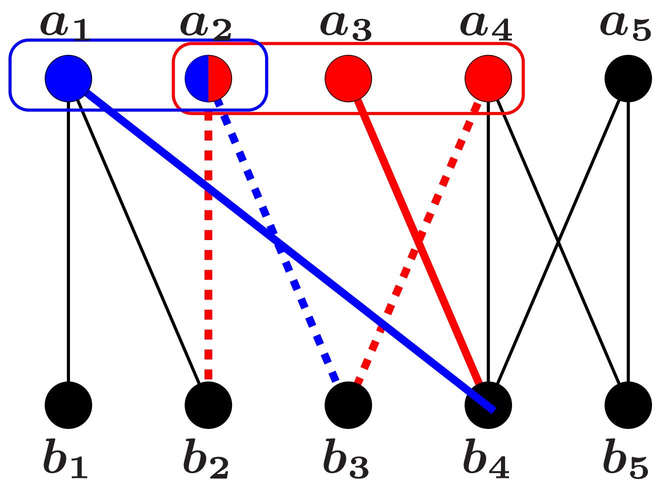

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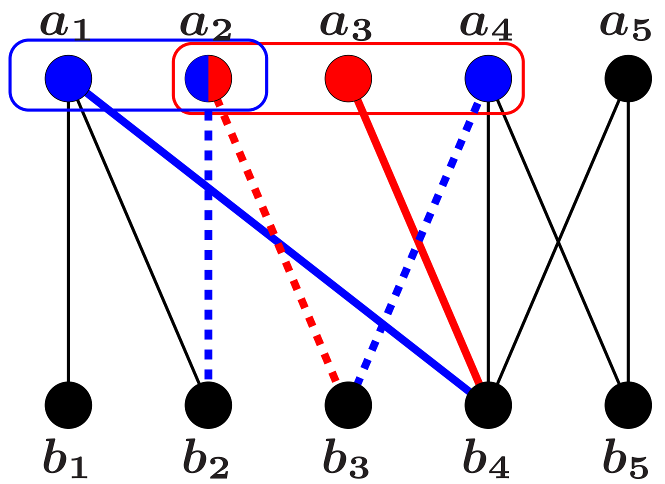

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Linear matroids

Fact: Let A be matrix and let E be the set of column vectors in A. The subsets $E' \subseteq E$ that are linearly independent form a matroid.

Proof:

(1) and (2) are clear.

(3) If $|X| > |Y|$ and both of them are linearly independent, then X spans a subspace with larger dimension than Y. Thus X contains a vector v not spanned by $Y \Rightarrow Y \cup \{v\}$ is linearly independent.

Example:

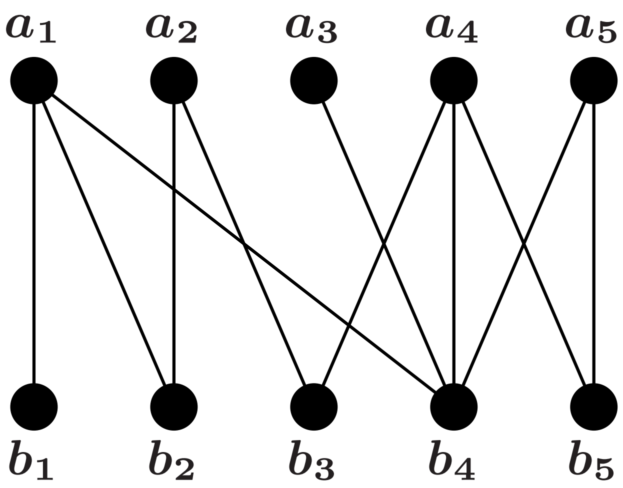
$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 6 \end{pmatrix} \end{array} \Rightarrow \mathcal{M} = \{\emptyset, a, b, c, d, ab, ac, ad, bc, bd\}$$

Representation

- If $\mathcal{M}$ is the matroid of the columns of a matrix A, then A is a **representation** of $\mathcal{M}$.
- If A is a matrix over a field $\mathbb{F}$, then $\mathcal{M}$ is **representable** over $\mathbb{F}$.
- If $\mathcal{M}$ is representable over some field $\mathbb{F}$, then $\mathcal{M}$ is **linear.**
- There are non-linear matroids (i.e., they cannot be represented over any field).

Transversal matroids are linear

Fact: Let $G(A, B; E)$ be a bipartite graph. Those subsets of A that can be covered by a matching form a linear matroid.



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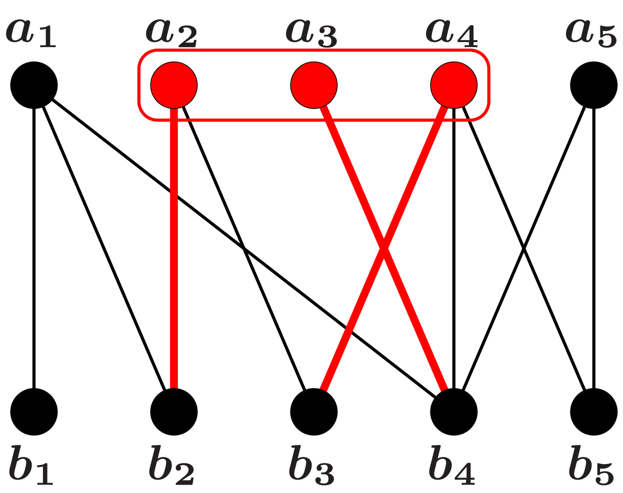

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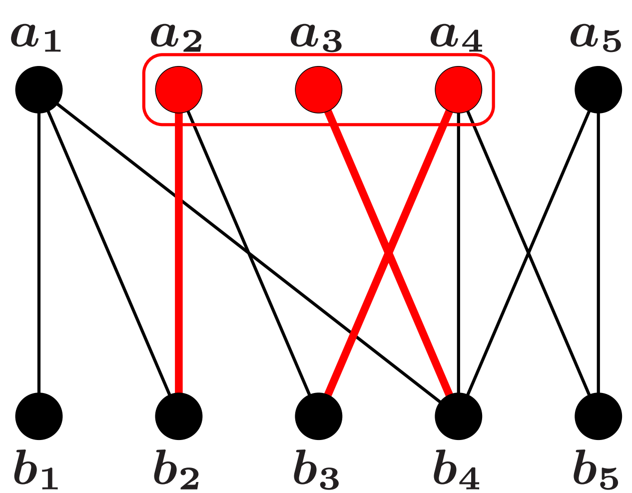

$$
\begin{array}{c c}
 & \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & a_5 \end{array} \\
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Construct the bipartite adjacency matrix: if a_i and b_j are neighbors, then the i-th element of row j is a random integer between 1 and N.

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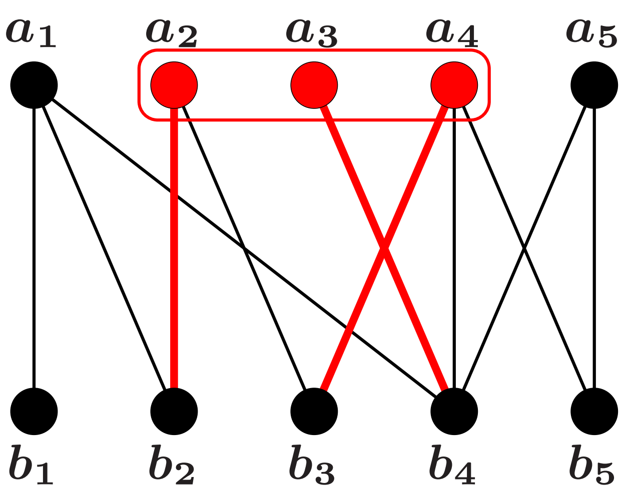

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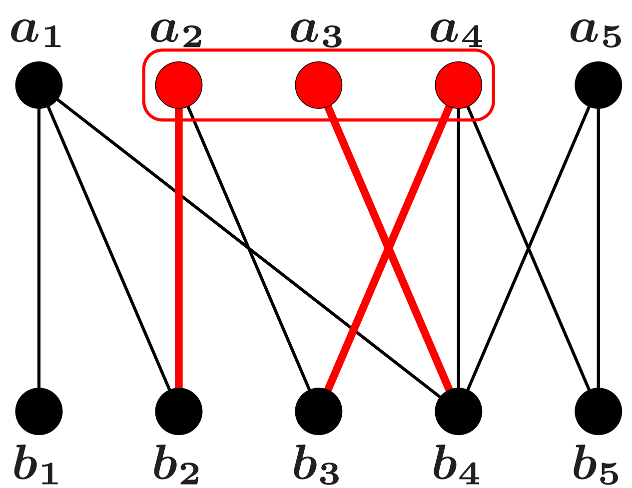

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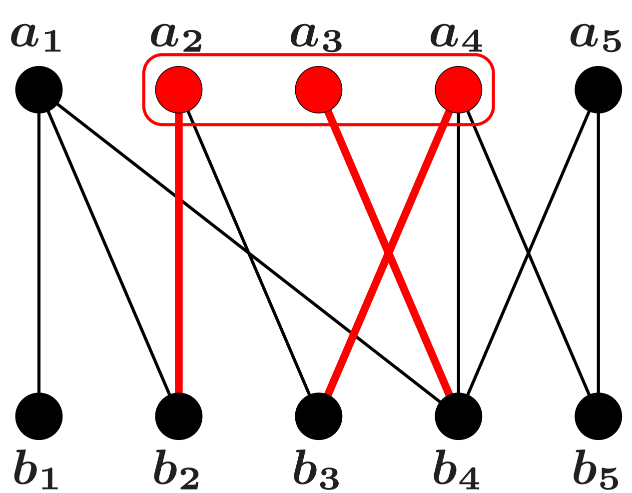

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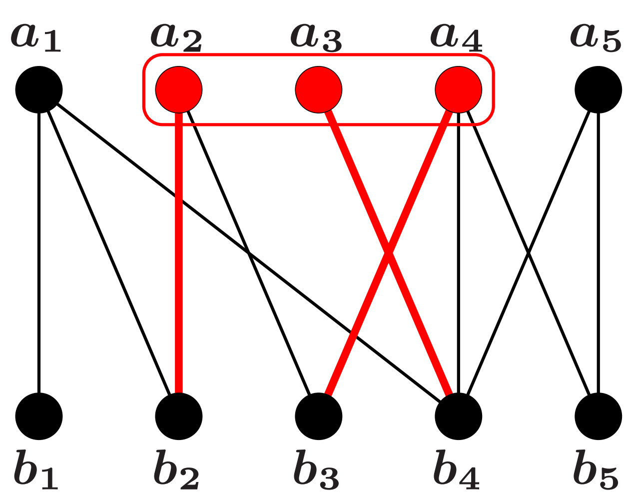

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Elements can be matched $\Rightarrow$ The determinant is nonzero with high probability (Schwartz-Zippel)

FPT result

Main result: Let $\mathcal{M}$ be a linear matroid over E, given by a representation A. Let $\mathcal{S}$ be a collection of subsets of E, each of size at most ℓ. It can be decided in randomized time $f(k,\ell) \cdot n^{O(1)}$ whether $\mathcal{M}$ has an independent set that is the union of k disjoint sets from $\mathcal{S}$.

Immediate application: k-DISJOINT TRIANGLES is (randomized) FPT (let $\mathcal{S}$ be the set of all triangles in the graph).

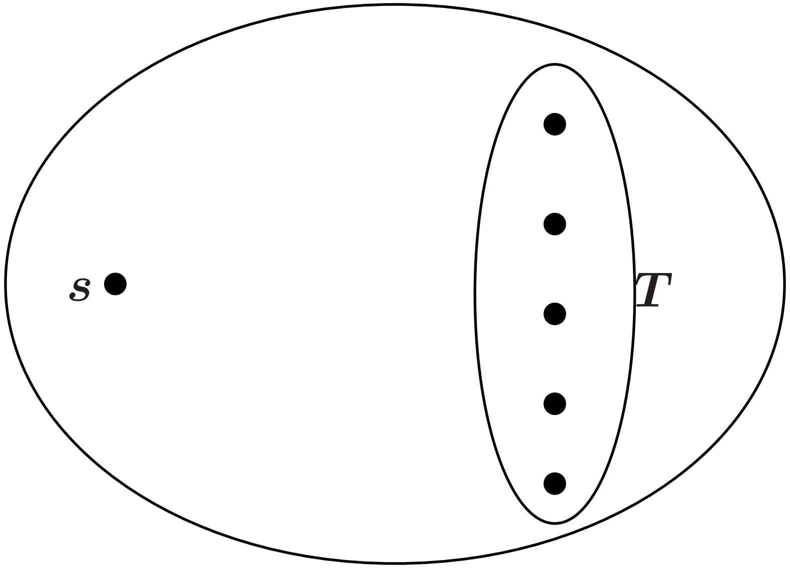
Two not so obvious applications:

- RELIABLE TERMINALS
- ASSIGNMENT WITH COUPLES

RELIABLE TERMINALS

Let D be a directed graph with a source vertex s and a subset T of vertices.

Task: Select k terminals $t_1, \ldots, t_k \in T$, and ℓ paths from s to each t_i such that these $k \cdot \ell$ paths are pairwise internally vertex disjoint.



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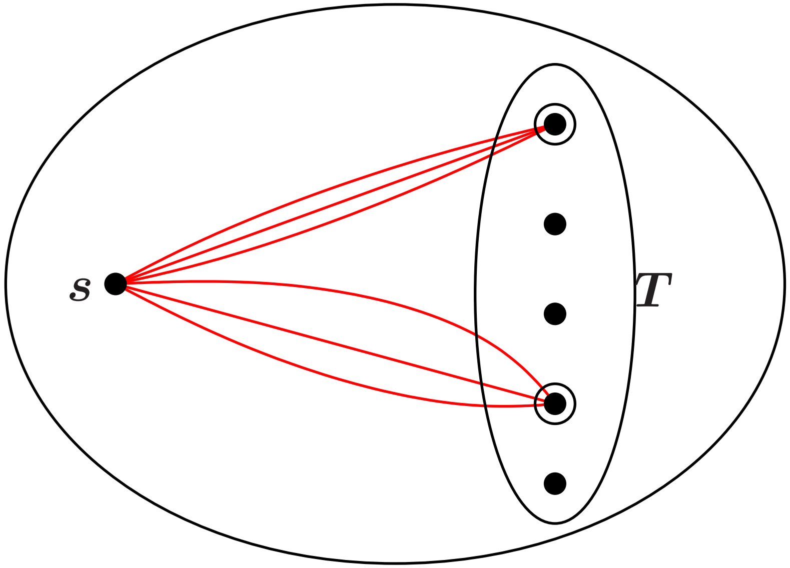

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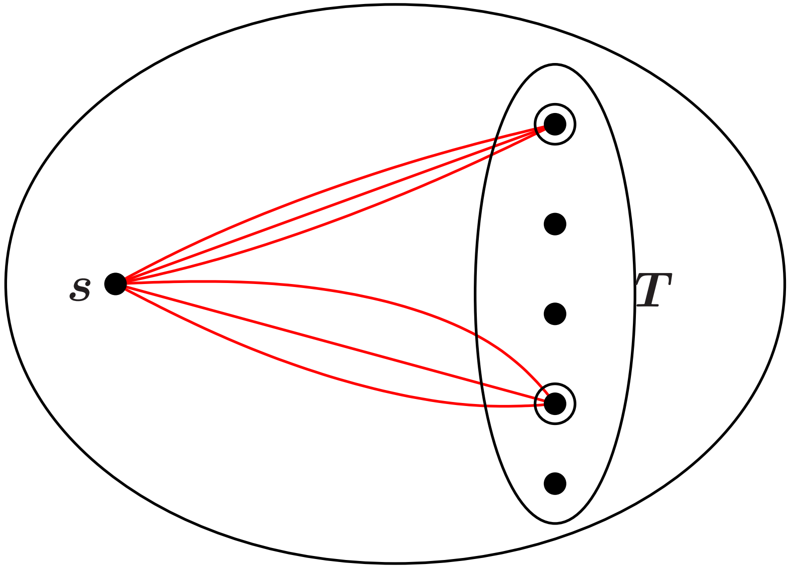

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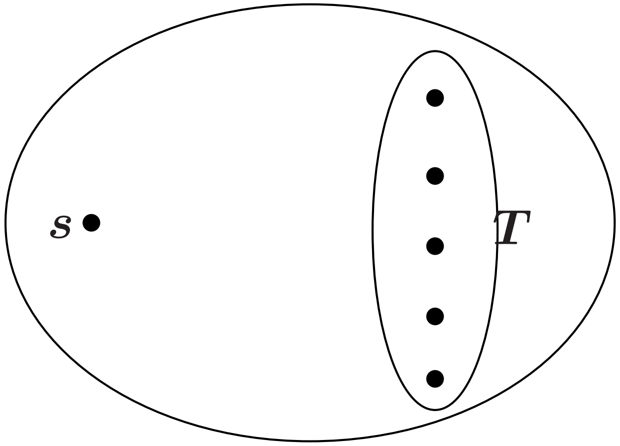
Theorem: The problem can be solved in randomized time $f(k, \ell) \cdot n^{O(1)}$.

Reliable terminals

A technical trick: replace each $t \in T$ with ℓ copies, and replace s with a set S of $k \cdot \ell$ copies.



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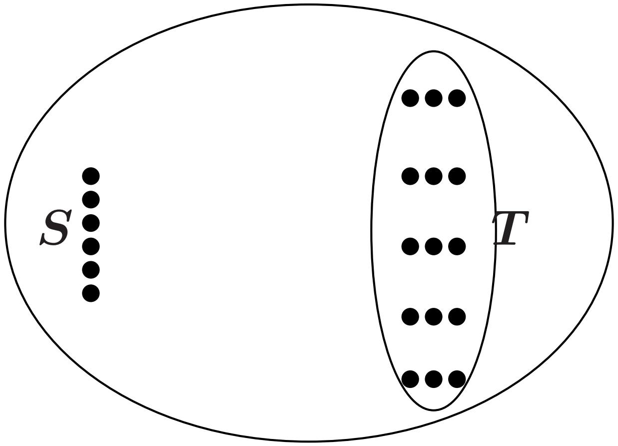

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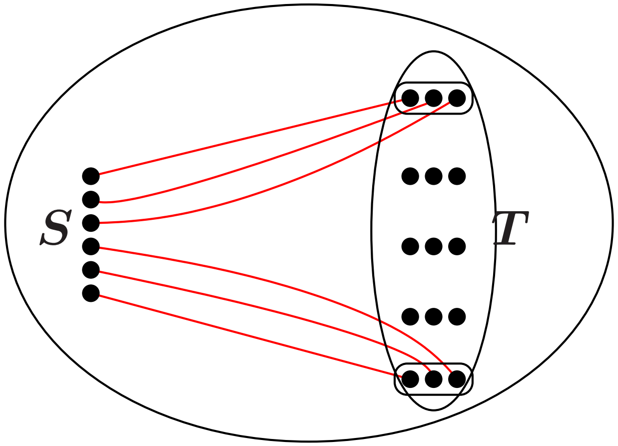

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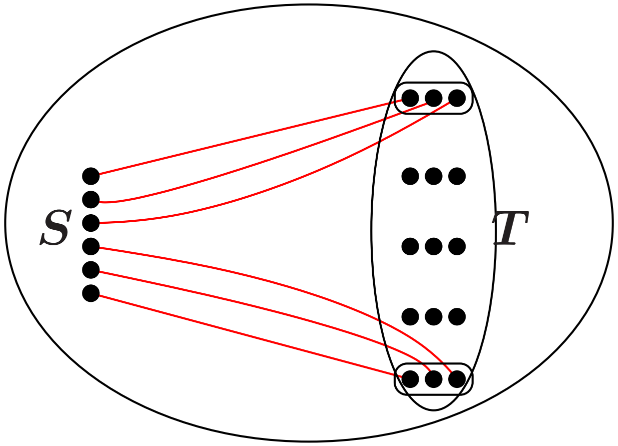

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Fact: [Perfect] Let D be a directed graph and S a subset of vertices. Those subsets X that can be reached from S by disjoint paths form a matroid.

RELIABLE TERMINALS

Fact: [Perfect] Let D be a directed graph and S a subset of vertices. Those subsets X that can be reached from S by disjoint paths form a matroid.

The problem is equivalent to finding k blocks whose union is independent in this matroid $\Rightarrow$ We can solve it in randomized time $f(k,\ell) \cdot n^{O(1)}$.

The matroid is actually a transversal matroid of an appropriately defined bipartite graph, hence it is linear and we can construct a representation for it.

ASSIGNMENT WITH COUPLES

Task: Assign people to jobs (bipartite matching).

However, the set of people includes couples and the members of a couple cannot be assigned independently (say, they want to be in the same town).

Task: Given

- a set of singles and a list of suitable jobs for each single,
- a set of couples and a list of suitable pairs of jobs for each couple,

assign a job to each single and a pair of jobs to each couple.

Theorem: ASSIGNMENT WITH COUPLES is randomized FPT parameterized by the number k of couples.

ASSIGNMENT WITH COUPLES

J: jobs, S: singles, C: couples

Let $X \subseteq J$ be in $\mathcal{M}$ if and only if S has a matching with $J \setminus X$.

Lemma: $\mathcal{M}$ is matroid.

Let $\mathcal{M}'$ be the matroid over $J \cup C$ such that $X \in \mathcal{M}' \Leftrightarrow X \cap J \in \mathcal{M}$.

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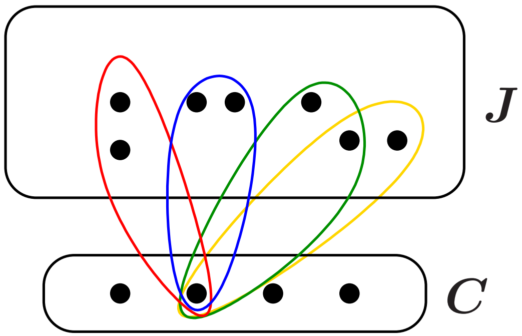

For each couple $c \in C$ and suitable pair $\{j_1, j_2\}$, add triple $\{c, j_1, j_2\}$ to $\mathcal{S}$.

The k couples and all the singles can be a assigned a job

$$\Updownarrow$$

There are k disjoint triples in $\mathcal{S}$ whose union is independent in $\mathcal{M}'$

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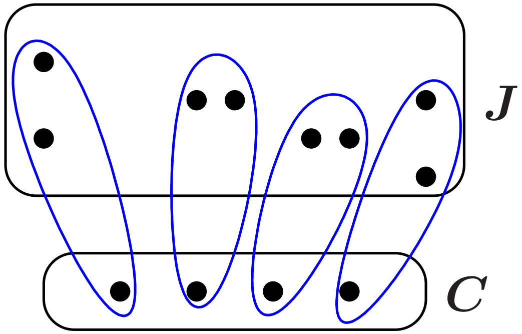

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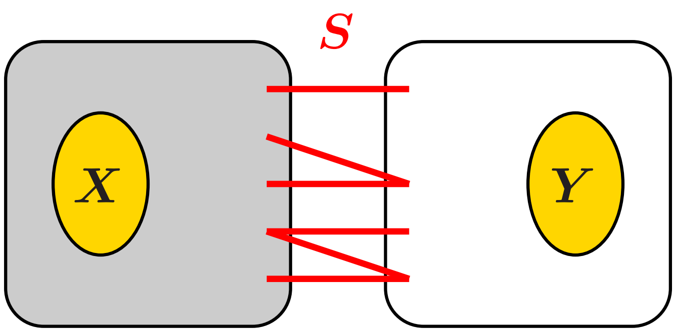
Cut problems

MULTIWAY CUT

Task: Given a graph G, a set T of vertices, and an integer k, find a set S of at most k edges that separates T (each component of $G \setminus S$ contains at most one vertex of T).

Polynomial for $|T| = 2$, but NP-hard for $|T| = 3$.

Theorem: MULTIWAY CUT is FPT parameterized by k.



$\delta(R)$: set of edges leaving R

$\lambda(X, Y)$: minimum number of edges in an (X, Y)-separator

Submodularity

Fact: The function δ is **submodular:** for arbitrary sets A, B,

$$|\delta(A)| \quad + \quad |\delta(B)| \quad \geq \quad |\delta(A \cap B)| \quad + \quad |\delta(A \cup B)|$$

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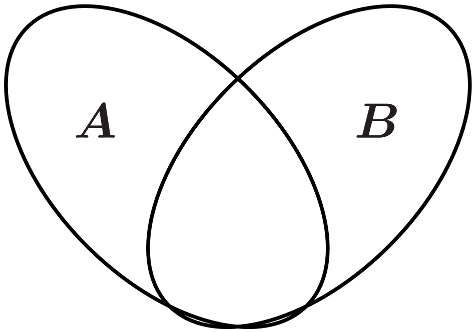
Proof: Determine separately the contribution of the different types of edges.



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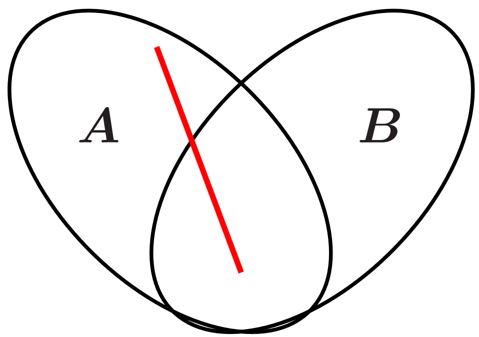

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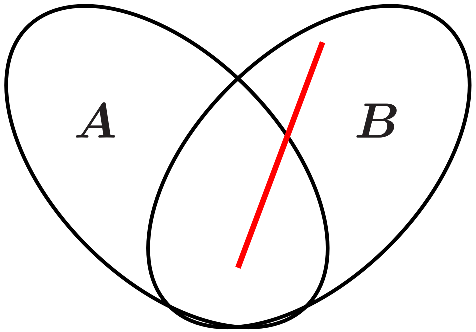

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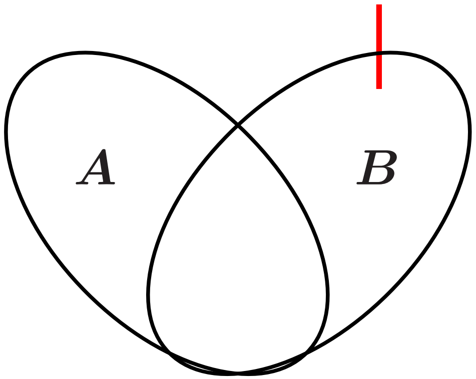

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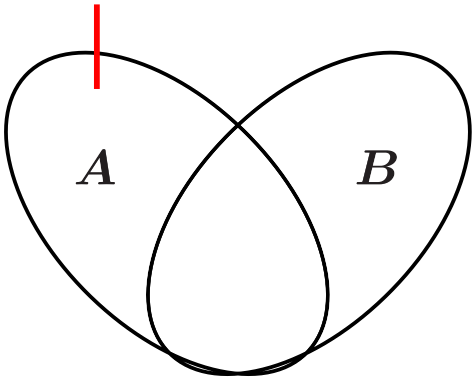

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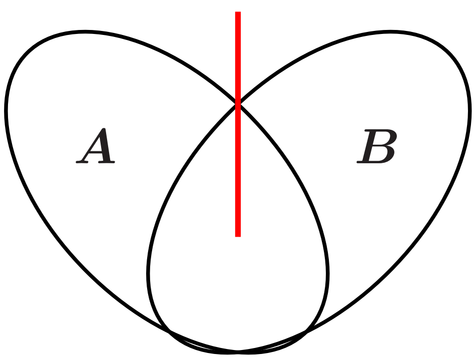

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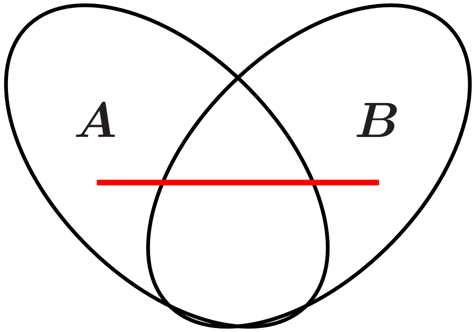
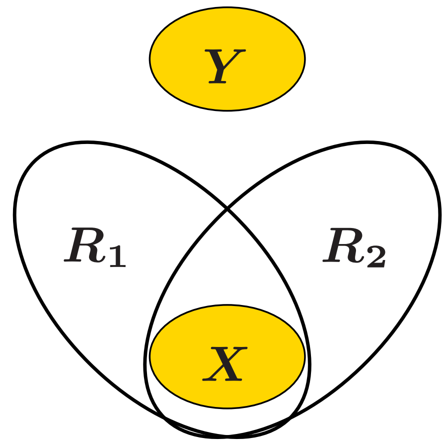


Submodularity

Consequence: There is a unique maximal $R_{\text{max}} \supseteq X$ such that $\delta(R_{\text{max}})$ is an (X, Y)-separator of size $\lambda(X, Y)$.

Proof: Let $R_1, R_2 \supseteq X$ be two sets such that $\delta(R_1), \delta(R_2)$ are (X, Y)-separators of size $\lambda := \lambda(X, Y)$.



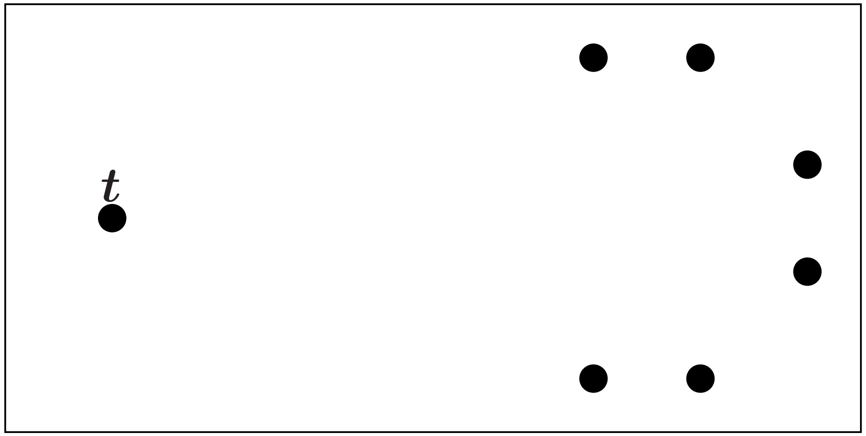
$$\begin{array}{ccccccccc} |\delta(R_1)| & + & |\delta(R_2)| & \geq & |\delta(R_1 \cap R_2)| & + & |\delta(R_1 \cup R_2)| \\ \lambda & & \lambda & & \geq \lambda & & \end{array}$$

$$\Rightarrow |\delta(R_1 \cup R_2)| \leq \lambda$$

Note: Analogous result holds for a unique minimal R_{min}.

MULTIWAY CUT

Intuition: Consider a $t \in T$. A subset of the solution separates t and $T \setminus \{t\}$.



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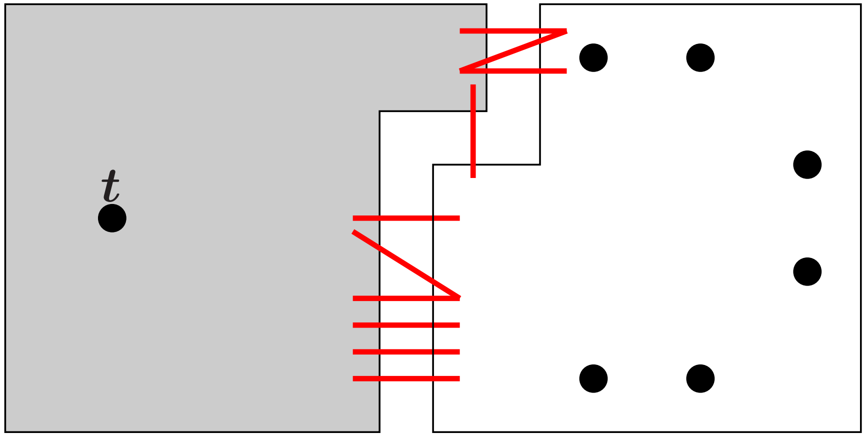

There are many such separators.

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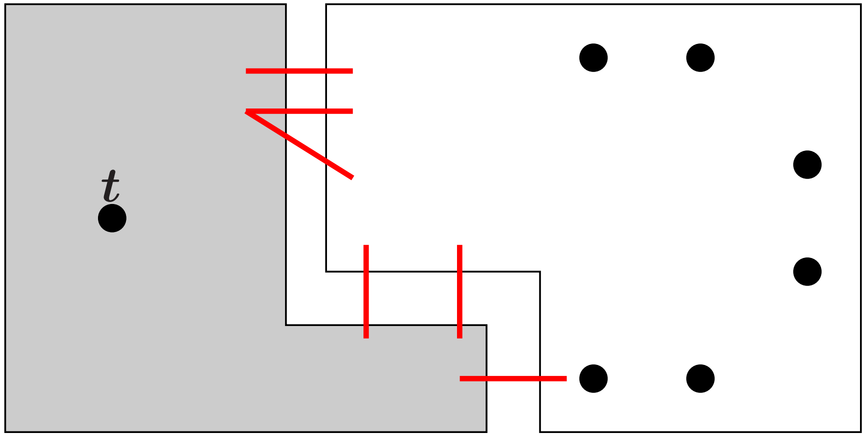

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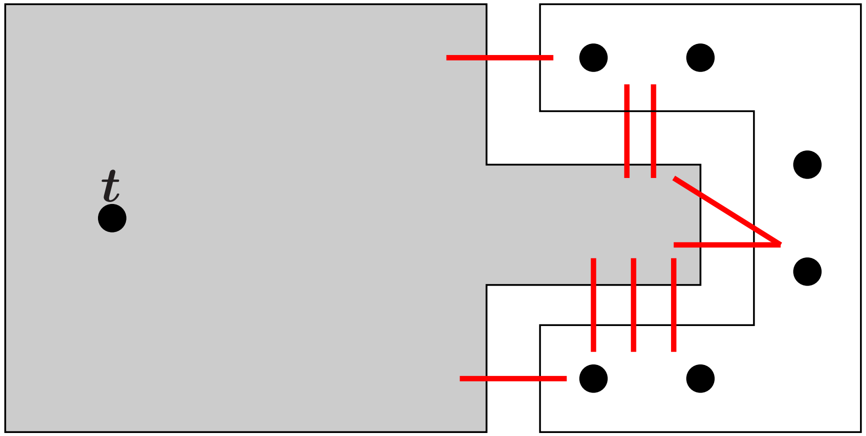
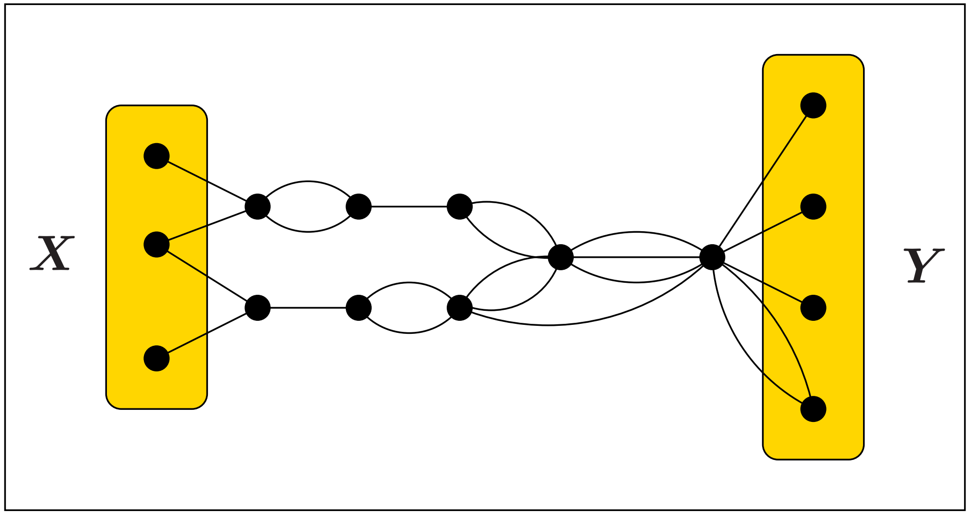


There are many such separators.

But a separator farther from t and closer to $T \setminus \{t\}$ seems to be more useful.

Important separators

Definition: An (X, Y)-separator $\delta(R)$ ($R \supseteq X$) is **important** if there is no (X, Y)-separator $\delta(R')$ with $R \subset R'$ and $|\delta(R')| \leq |\delta(R)|$.



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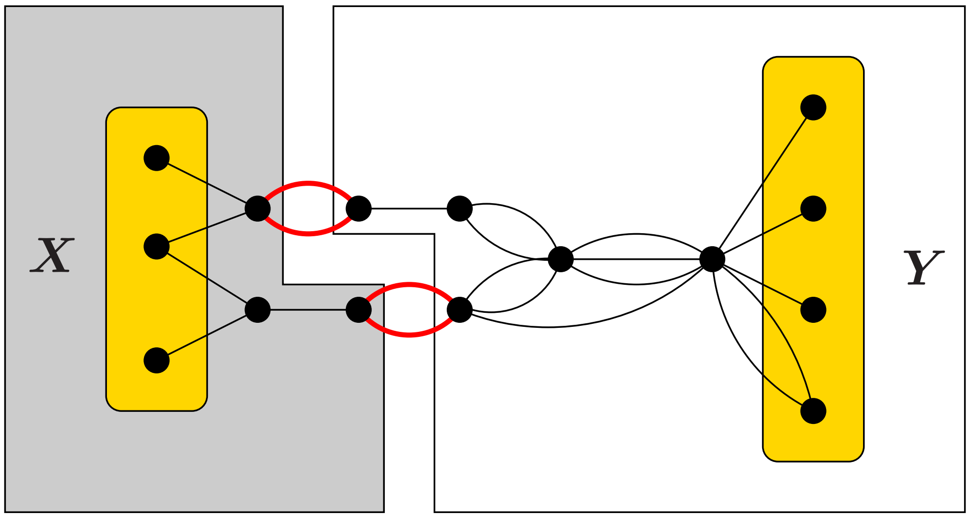

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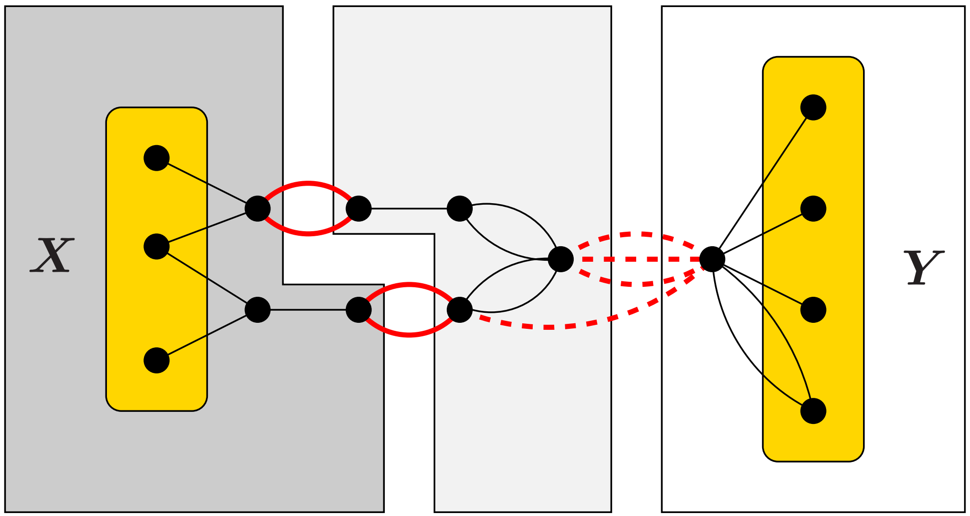

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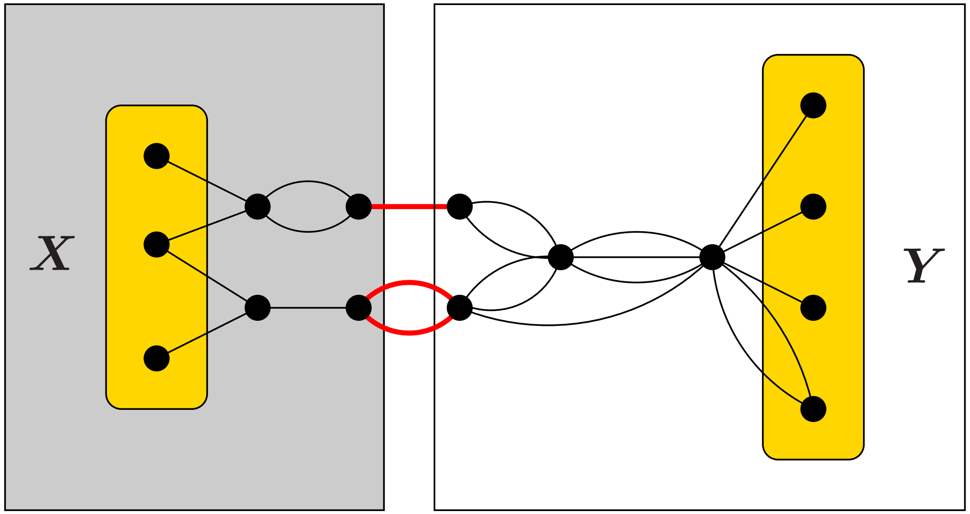

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Lemma: Let $t \in T$. The MULTIWAY CUT problem has a solution S such that S contains an important $(t, T \setminus \{t\})$-separator.

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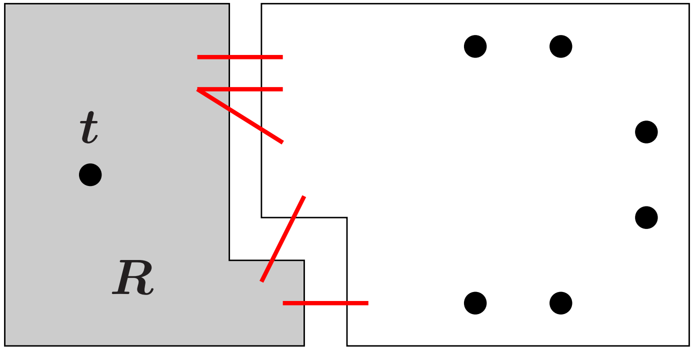
Proof: Let R be the vertices reachable from t in $G \setminus S$.



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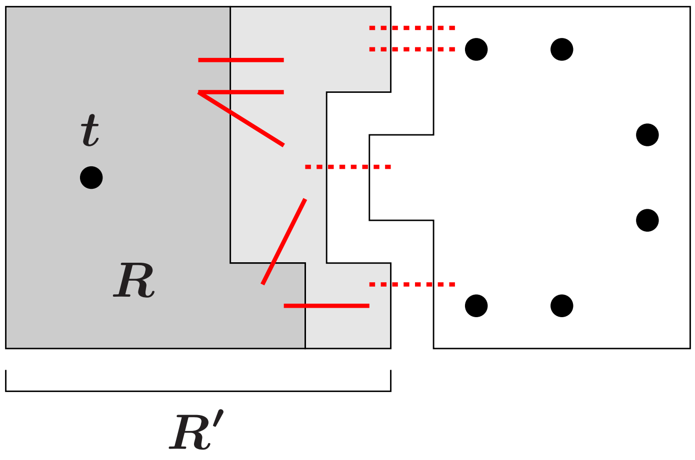

If $\delta(R)$ is not important, then there is an important separator $\delta(R')$ that dominates it. Replace S with $S' := (S \setminus \delta(R)) \cup \delta(R')$ ($|S'| \leq |S|$).

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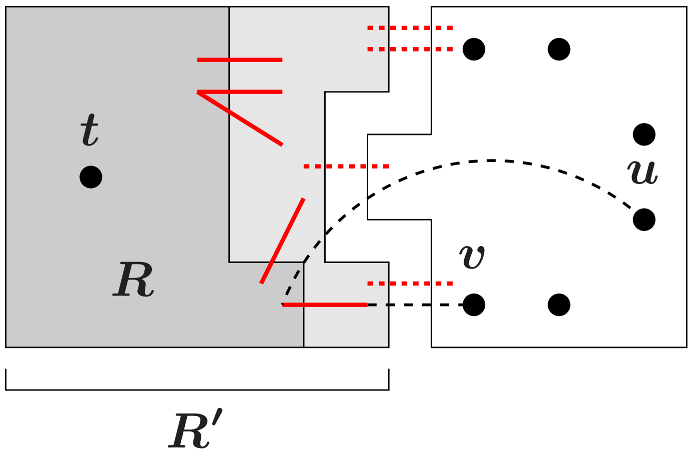

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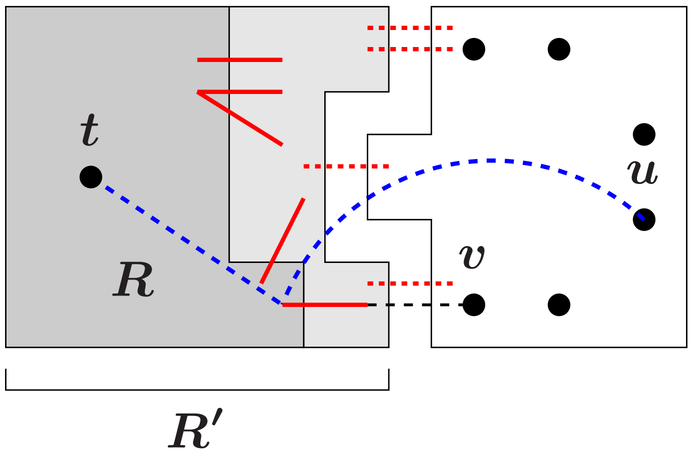

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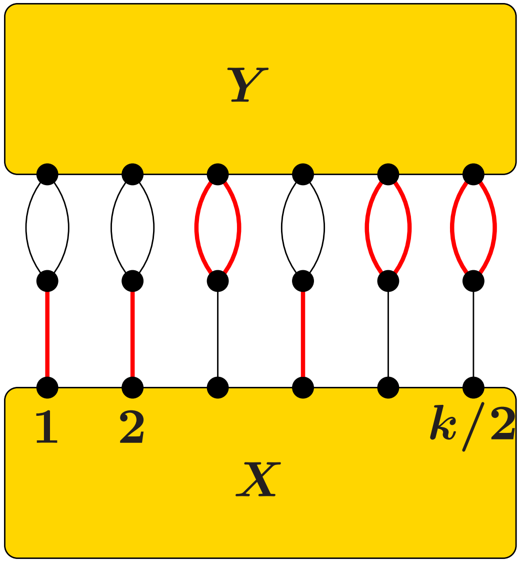
Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Example:



There are exactly $2^{k/2}$ important (X, Y)-separators of size at most k in this graph.

Important separators

Lemma: There are at most 4^k important (X, Y)-separators of size at most k.

Proof: First we show that $R_{\text{max}} \subseteq R$ for every important separator $\delta(R)$.

$$\underbrace{|\delta(R_{\text{max}})|}_{\lambda} + |\delta(R)| \geq \underbrace{|\delta(R_{\text{max}} \cap R)|}_{\geq \lambda} + |\delta(R_{\text{max}} \cup R)|$$

$$\Downarrow$$

$$|\delta(R_{\text{max}} \cup R)| \leq |\delta(R)|$$

$$\Downarrow$$

If $R \neq R_{\text{max}} \cup R$, then $\delta(R)$ is not important.

Thus the important (X, Y)- and (R_{max}, Y)-separators are the same.

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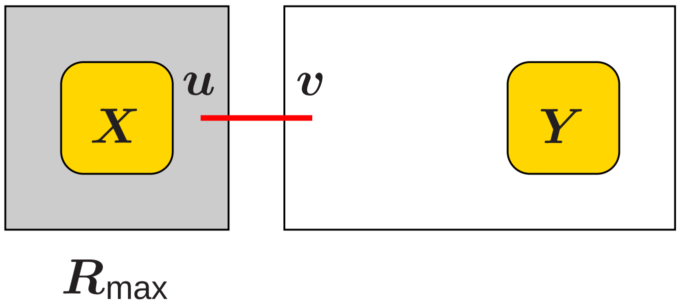


The edge uv leaving R_{max} is either in the separator or not.

Branch 1: Edge uv is in the separator. Delete uv and set $k := k - 1$.
$\Rightarrow$ k decreases by one, λ decreases by at most 1.
Branch 2: Edge uv is not in the separator. Set $X := R_{\text{max}} \cup \{v\}$.
$\Rightarrow$ k remains the same, λ increases by 1.

The measure $2k - \lambda$ decreases in each step.
$\Rightarrow$ Height of the search tree $\le 2k \Rightarrow \le 2^{2k}$ important separators.

Algorithm for MULTIWAY CUT

1. If every vertex of T is in a different component, then we are done.
2. Let $t \in T$ be a vertex with that is not separated from every $T \setminus \{t\}$.
3. Branch on a choice of an important $(\{t\}, T \setminus \{t\})$ separator S of size at most k.
4. Set $G := G \setminus S$ and $k := k - |S|$.
5. Go to step 1.

Size of the search tree:

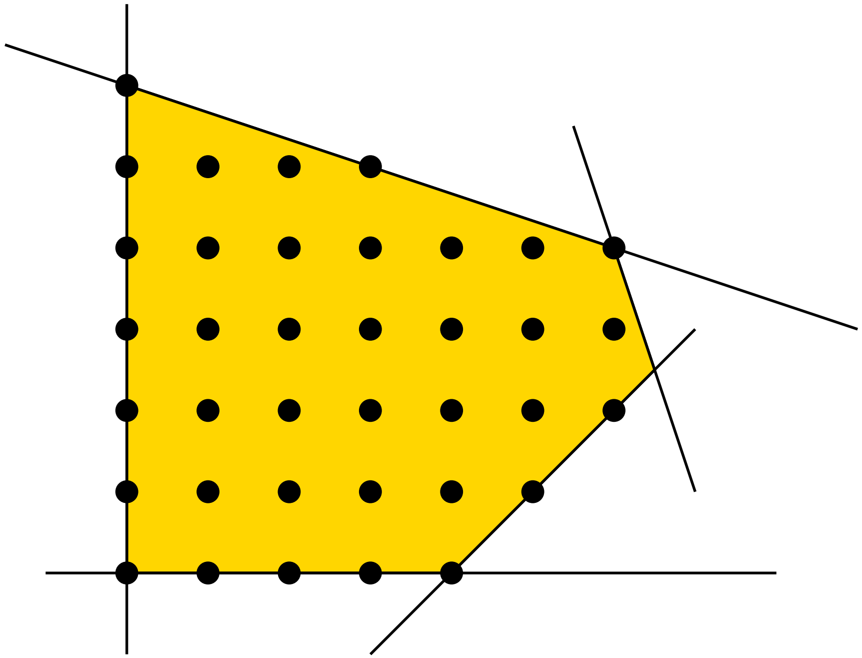
- When searching for the important separator, $2k - \lambda$ decreases at each branching.
- When choosing the next t, λ changes from 0 to positive, thus $2k - \lambda$ does not increase.

Size of the search tree is at most 2^{2k}.

Other separation problems

- Some other variants:
  - $|T|$ as a parameter
  - MULTITERMINAL CUT: pairs $(s_1, t_1), \ldots, (s_\ell, t_\ell)$ have to be separated.
  - Directed graphs
  - Planar graphs
- Useful for deletion-type problems such as DIRECTED FEEDBACK VERTEX SET (via iterative compression).
- Important separators: is it relevant for a given problem?

Integer Linear Programming



Integer Linear Programming

Linear Programming (LP): important tool in (continuous) combinatorial optimization. Sometimes very useful for discrete problems as well.

$$\max c_1 x_1 + c_2 x_2 + c_3 x_3$$

s.t.

$$\begin{aligned} x_1 + 5x_2 - x_3 &\leq 8 \\ 2x_1 - x_3 &\leq 0 \\ 3x_2 + 10x_3 &\leq 10 \\ x_1, x_2, x_3 &\in \mathbb{R} \end{aligned}$$

Fact: It can be decided if there is a solution (feasibility) and an optimum solution can be found in polynomial time.

Integer Linear Programming

Integer Linear Programming (ILP): Same as LP, but we require that every x_i is integer.

Very powerful, able to model many NP-hard problems. (Of course, no polynomial-time algorithm is known.)

Theorem: ILP with p variables can be solved in time $p^{O(p)} \cdot n^{O(1)}$.

CLOSEST STRING

Task: Given strings $s_1, \ldots, s_k$ of length L over alphabet Σ, and an integer d, find a string s (of length L) such that $d(s, s_i) \leq d$ for every $1 \leq i \leq k$.

Note: $d(s, s_i)$ is the Hamming distance.

Theorem: CLOSEST STRING parameterized by k is FPT.

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Theorem: CLOSEST STRING is NP-hard for $\Sigma = \{0, 1\}$.

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Closest String

An instance with $k = 5$ and a solution for $d = 4$:

s_1	CBDCCACBB
s_2	ABDBCABDB
s_3	CDDBACCBD
s_4	DDABACCBD
s_5	ACDBDDCBC
	ADDBCACBD

Each column can be described by a partition $\mathcal{P}$ of $[k]$.

The instance can be described by an integer $c_{\mathcal{P}}$ for each partition $\mathcal{P}$: the number of columns with this type.

Closest String

An instance with $k = 5$ and a solution for $d = 4$:

s_1	**CB**D**C**CACB**B**
s_2	A**B**DBCA**BDB**
s_3	**C**DDB**AC**CBD
s_4	**D**D**A**B**AC**CBD
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Describing a solution: If C is a class of $\mathcal{P}$, let $x_{\mathcal{P},C}$ be the number of type $\mathcal{P}$ columns where the solution agrees with class C.

There is a solution iff the following ILP has a feasible solution:

$$\sum_{C \in \mathcal{P}} x_{\mathcal{P},C} \le c_{\mathcal{P}} \qquad \forall \text{partition } \mathcal{P}$$

$$\sum_{i \notin C, C \in \mathcal{P}} x_{\mathcal{P},C} \le d \qquad \forall 1 \le i \le k$$

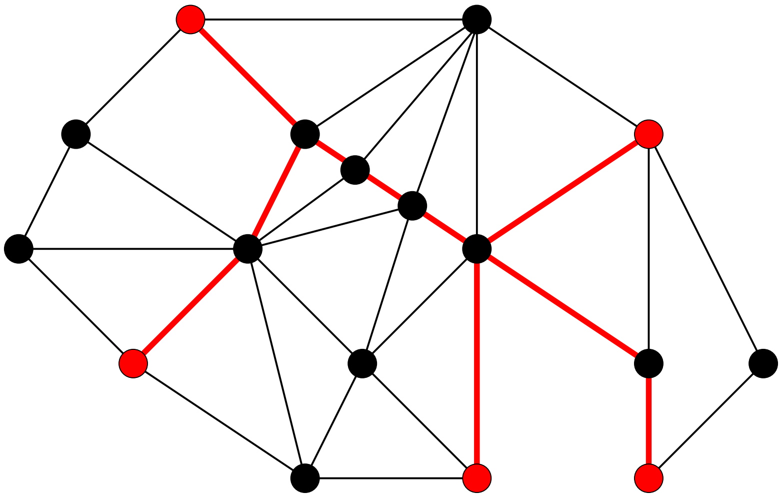
$$x_{\mathcal{P},C} \ge 0 \qquad \forall \mathcal{P}, C$$

Number of variables is $\le B(k) \cdot k$, where $B(k)$ is the no. of partitions of $[k]$
$\Rightarrow$ The ILP algorithm solves the problem in time $f(k) \cdot n^{O(1)}$.

STEINER TREE

STEINER TREE

Task: Given a graph G with weighted edges and a set S of k vertices, find a tree T of minimum weight that contains S.



Known to be NP-hard. For fixed k, we can solve it in polynomial time: we can guess the Steiner points and the way they are connected.

Theorem: STEINER TREE is FPT parameterized by $k = |S|$.

STEINER TREE

Solution by dynamic programming. For $v \in V(G)$ and $X \subseteq S$,

$c(v, X) :=$ minimum cost of a Steiner tree of X that contains v

$d(u, v) :=$ distance of u and v

Recurrence relation:

$$c(v, X) = \min_{\substack{u \in V(G) \\ \emptyset \subset X' \subset X}} c(u, X' \setminus u) + c(u, (X \setminus X') \setminus u) + d(u, v)$$

Steiner Tree

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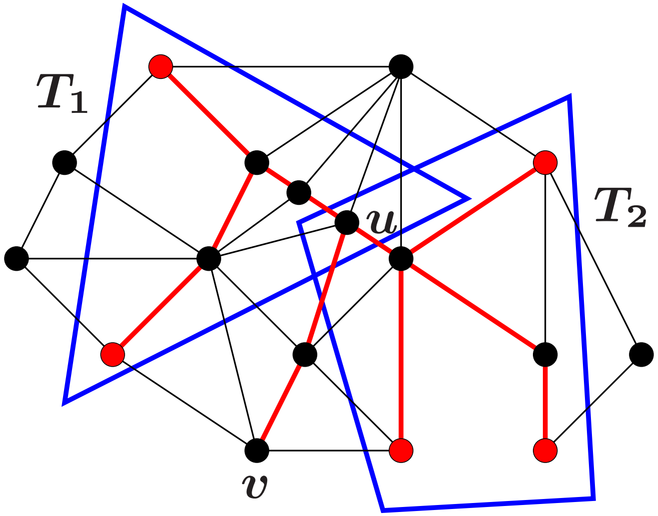
- $\leq$: A tree T_1 realizing $c(u, X' \setminus u)$, a tree T_2 realizing $c(u, (X \setminus X') \setminus u)$, and the path uv gives a (superset of a) Steiner tree of X containing v.



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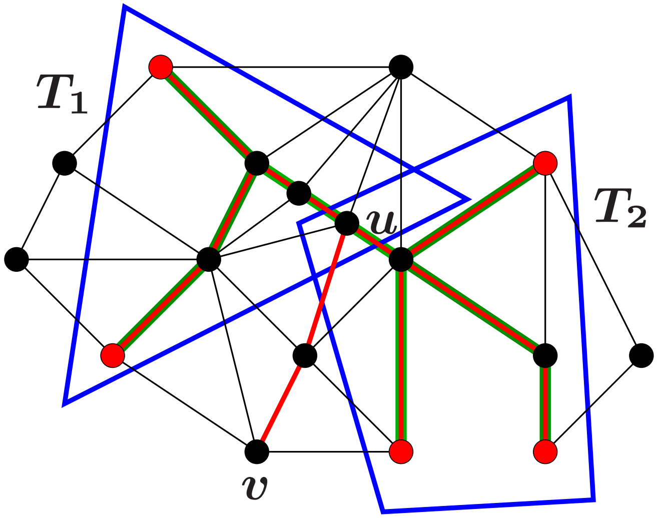
- $\geq$: Suppose T realizes $c(v, X)$, let T' be the minimum subtree containing X. Let u be a vertex of T' closest to v. If $|X| > 1$, then there is a component C of $T \setminus u$ that contains a subset $\emptyset \subset X' \subset X$ of terminals. Thus T is the disjoint union of a tree containing $X' \setminus u$ and u, a tree containing $(X \setminus X') \setminus u$ and u, and the path uv.



STEINER TREE

Recurrence relation:

$$c(v,X) = \min_{\substack{u \in V(G) \\ \emptyset \subset X' \subset X}} c(u, X' \setminus u) + c(u, (X \setminus u) \setminus X') + d(u,v)$$

Running time:

$2^k|V(G)|$ variables $c(v,X)$, determine them in increasing order of $|X|$.
Variable $c(v,X)$ can be determined by considering $2^{|X|}$ cases. Total number of cases to consider:

$$\sum_{X \subseteq T} 2^{|X|} = \sum_{i=1}^{k} \binom{k}{i} 2^i \leq (1+2)^k = 3^k.$$

Running time is $O^*(3^k)$.

Note: Running time can be reduced to $O^*(2^k)$ with clever techniques.

Conclusions

- Many nice techniques invented so far — and probably many more to come.
- A single technique might provide the key for several problems.
- How to find new techniques? By attacking the open problems!
- Needed: flexible, highly expressive problems. Solve other problems by reduction to these problems.
  - Courcelle's Theorem
  - The matroid result
  - 2SAT DELETION: given a 2SAT formula and an integer k, delete k clauses to make it satisfiable
  - Constraint Satisfaction Problems